# Differential Equations

## Lecture Set 01
## Course Overview and
## Introduction to Differential Equations

林惠勇
Huei-Yung Lin

lin@ee.ccu.edu.tw

Robot Vision Lab
Department of Electrical Engineering
National Chung Cheng University
Chiayi 621, Taiwan

# Course Overview

- Instructor
  - 林惠勇, lin@ee.ccu.edu.tw
  - Office Hour: Wed. 10:00 – 12:00, or by appointment
  - Room 431A
  - Phone: (05) 272-0411 ext. 33224
- Class Time and Place
  - Tu, Th: 13:00 – 14:30 (originally 13:15 – 14:30)
  - Room 127
- Teaching Assistant
  - Office Hours: 8 hours total
  - Room 122
  - Phone: (05) 272-0411 ext. 23274

# Course Overview

- Course Webpage
  - School's ecourse website
  - The lecture slides will be posted on the course website.
  - Old lecture slides (from the previous year) are also available on the ecourse website.
- Textbook:
  - Differential Equations with Boundary-Value Problems, Metric Version, **9th Edition** by Dennis G. Zill, ISBN: 978-1-337-55988-1, CENGAGE Learning
  - Differential Equations with Boundary-Value Problems, Metric Version, **8th** Edition, by Zill and Wright, ISBN: 987-1-305-97063-2, CENGAGE Learning
- Reference:
  - Differential Equations, 3rd Edition, by S. L. Ross, ISBN: 978-0471032946, John Wiley & Sons

# Rules – How To Survive This Course?

- Exams – 75%
  - Three exams – 25% each
  - One optional final exam – 25% (comprehensive)
- Homeworks – 5%
  - No late homework
  - Only 2 problems will be graded (More will be discussed later.)
- Quizzes and participation – 20%
  - Several in-class quizzes (9/24, 11/7, 12/12, to be confirmed)
  - Most problems from homework exercises and textbook examples
- Important dates: (TBC)
  - Exam I: 10/8, 13:00 – 14:30
  - Exam II: 11/26, 13:00 – 14:30
  - Exam III: 1/2, 13:00 – 14:30
  - Exam IV (optional, comprehensive): 1/9, 13:00 – 14:30

# Rules – How To Survive This Course?

- Extra lectures:
  - 13:00 – 13:15, 15 minutes before lectures, starting this Thursday
- Absolutely NO makeup exams!!!
  - Exam IV can fill one skipped exam.
- Do NOT skip classes.
- Take notes in class, not everything will appear on lecture slides.
- Do the homework by yourself.

Questions?

# Course Objective

- To familiarize students with theories of differential equations and problem-solving techniques.
- The goal of the course is to train the students with capabilities of building mathematical models for the physical systems and solve them in the time domain.

# Course Topics

- Introduction to Differential Equations
- First-Order Differential Equations
- Higher-Order Differential Equations
- Series Solutions of Linear Equations
- The Laplace Transform
- Systems of Linear First-Order Differential Equations
- Numerical Solutions of Ordinary Differential Equations
- Fourier Series
- Boundary-Value Problems in Rectangular Coordinates
- Integral Transforms

# Definition 1.1.1: Differential Equation

## Definition (1.1.1: Differential Equation)

An equation containing the derivatives of one or more dependent variables, with respect to one or more independent variables, is said to be a **differential equation (DE)**.

## Example

The equation

$$\frac{dy}{dx} = 0.2xy$$

is a differential equation and its solution is

$$y = e^{0.1x^2}$$

Note that the above equation is differentiable on the interval $(-\infty, \infty)$.

# Example

## Example

The equation

$$\frac{dy}{dx} = 0.2xy$$

is a differential equation and its solution is

$$y = e^{0.1x^2}$$

# Classification of Differential Equations

The differential equations can be classified by **type**, **order**, and **linearity**.

- Classification by Type:
  - Ordinary differential equation (ODE):

$$\frac{dy}{dx} + 5y = e^x, \qquad \frac{d^2y}{dx^2} - \frac{dy}{dx} + 6y = 0, \qquad \frac{dx}{dt} + \frac{dy}{dt} = 2x + y$$

  - Partial differential equation (PDE)

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \qquad \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} - 2\frac{\partial u}{\partial t}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

  - ODE has derivatives with respect to one independent variable, but PDE has derivatives with respect to several independent variables.

# Classification of Differential Equations

- Classification by Order:
  - First order differential equation

$$\frac{dy}{dx} + 5y = e^x, \qquad M(x,y)dx + N(x,y)dy = 0, \qquad \frac{dy}{dx} = f(x,y)$$

  - Second order differential equation

$$\frac{d^2y}{dx^2} + 5\left(\frac{dy}{dx}\right)^3 - 4y = e^x, \qquad \frac{d^2y}{dx^2} = f(x,y,y')$$

  - $n$th order differential equation

$$F(x,y,y',y'',\cdots,y^{(n)}) = 0, \qquad \frac{d^ny}{dx^n} = f(x,y,y',y'',\cdots,y^{(n-1)})$$

    The second one is referred to as the **normal form** of the first one.
  - Named by the highest order of derivatives involved.

# Classification of Differential Equations

- Classification by Linearity:
  - Linear differential equation

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = g(x)$$

$$(y-x)dx + 4xdy = 0, \qquad y'' - 2y' + y = 0, \qquad \frac{d^3 y}{dx^3} + x\frac{dy}{dx} - 5y = e^x$$

  - Nonlinear differential equation

$$(1-y)y' + 2y = e^x, \qquad \frac{d^2 y}{dx^2} + \sin y = 0, \qquad \frac{d^4 y}{dx^4} + y^2 = 0$$

  - Check the linearity of the dependent variables. (The linear combination of dependent variable's differentiation only. That is, $1, y, y', y'', \cdots$.)

# Definition 1.1.2: Solution of an ODE

## Definition (1.1.2: Solution of an ODE)

Any function $\phi$, defined on an interval $I$ and possessing at least $n$ derivatives that are continuous on $I$, which when substituted into an $n$th-order ODE reduces the equation to an identity, is said to be a **solution** of the equation on the interval.

In other words, a solution of an $n$th ODE

$$F(x, y, y', y'', \cdots, y^{(n)}) = 0$$

is a function $\phi$ that possesses at least $n$ derivatives and for which[1]

$$F(x, \phi(x), \phi'(x), \phi''(x), \cdots, \phi^{(n)}(x)) = 0 \qquad \text{for all } x \text{ in } I$$

[1] That is, $y = \phi(x)$, ...

# Interval of Definition

The interval $I$ in the definition is variously called the **interval of definition**, the **interval of existence** and can be an open interval $(a, b)$, a close interval $[a, b]$, an infinite interval $(a, \infty)$, and so on.

# Example 5: Verification of a Solution

Verify that the indicated function is a solution of the given DE on the inverval $(-\infty, \infty)$.

(a) $\dfrac{dy}{dx} = xy^{1/2}; \quad y = \dfrac{1}{16}x^4$

(b) $y'' - 2y' + y = 0; \quad y = xe^x$

# Solution Curve

The graph of a solution $\phi$ of an ODE is called a **solution curve**.

## Example

The graph $\phi(x) = \frac{1}{16}x^4$ is a solution curve of $dy/dx = xy^{1/2}$ from the previous example.

Since $\phi$ is a differentiable function, it is continuous on its interval $I$ of definition.

Thus, there may be a difference between the graph of the function $\phi$ and the graph of the solution $\phi$.

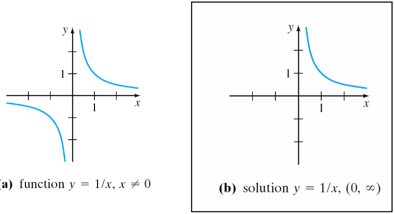
The domain of the function $\phi$ need not be the same as the interval $I$ of definition (or domain) of the solution $\phi$.

# Example: Function versus Solution

- The domain of $y = 1/x$, considered as a function, is the set of all real numbers $x$ except $0$. A plot of the function is shown in (a).

---

- The function $y = 1/x$ is also a solution of the linear 1st-order differential equation $xy' + y = 0$. (Verify this!)
- When we say $y = 1/x$ is a solution, we mean it is a function defined on $I$ on which it is differentiable and satisfies the DE. Thus, $y = 1/x$ is a solution of the DE on any interval not containing $0$.




**(a)** function $y = 1/x$, $x \neq 0$

**(b)** solution $y = 1/x$, $(0, \infty)$

# Explicit Solution

A solution in which the dependent variable is expressed solely in terms of the independent variable and constants is said to be an **explicit solution**.

### Example

$$y = \phi(x), \qquad y = \frac{1}{16}x^4, \qquad y = xe^x, \qquad y = 1/x$$

# Definition 1.1.3: Implicit Solution of an ODE

## Definition (1.1.3: Implicit Solution of an ODE)

A relation $G(x, y) = 0$ is said to be an **implicit solution** of an ODE

$$F(x, y, y', y'', \cdots, y^{(n)}) = 0$$

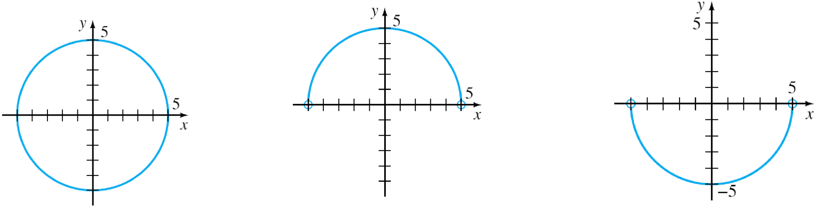
on an interval $I$, provided there exists <u>at least one</u> function $\phi$ that satisfies the relation as well as the differential equation on $I$.

# Example 7: Verificaion of an Implicit Solution

The relation $x^2 + y^2 = 25$ is an implicit solution of the DE

$$\frac{dy}{dx} = -\frac{x}{y}$$

on the interval $-5 < x < 5$.




**(a)** implicit solution $x^2 + y^2 = 25$

**(b)** explicit solution $y_1 = \sqrt{25 - x^2}$, $-5 < x < 5$

**(c)** explicit solution $y_2 = -\sqrt{25 - x^2}$, $-5 < x < 5$

In this case, $G(x, y) = x^2 + y^2 - 25$, $F(x, y, y', y'', \cdots, y^{(n)}) = y' + x/y$.

# Families of Solutions

- When solving a 1st-order DE

$$F(x, y, y') = 0$$

  we usually obtain a solution containing an arbitrary constant $c$. A solution containing an arbitrary constant represents a set $G(x, y, c) = 0$ of solutions called a **one-parameter family of solutions.** (Check $y' = y, y = ce^x$)

- When solving an $n$th-order DE

$$F(x, y, y', \cdots, y^{(n)}) = 0$$

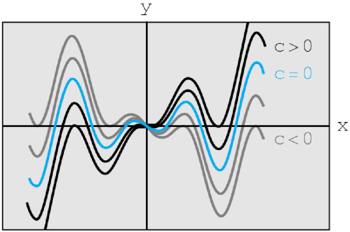
  we seek an **n-parameter family of solutions** $G(x, y, c_1, c_2, \cdots, c_n) = 0$.
  This means that a single DE can possess an infinite number of solutions corresponding to the unlimited number of choices for the parameter(s).

# Families of Solutions

A solution without any arbitrary parameters is called a **particular solution.**

## Example

An explicit solution of $xy' - y = x^2 \sin x$ is given by $y = cx - x\cos x$. The solution $y = -x\cos x$ is a particular solution. (Verify this!)



A DE may also possess a **singular solution**, which cannot be obtained by specializing any of the parameters in the family of solutions.

## Example

The relation $y = 0$ is a solution of the DE $dy/dx = xy^{1/2}$.
It cannot be obtained from the solution $y = (\frac{1}{4}x^2 + c)^2$.
Thus, the trivial solution $y = 0$ is a singular solution. (Verify this!)

# System of Differential Equations

A **system of ordinary differential equations** is two or more equations involving the derivatives of two or more unknown functions of a single independent variable.

## Example

A system of two first-order differential equations is given by

$$\frac{dx}{dt} = f(t, x, y)$$
$$\frac{dy}{dt} = g(t, x, y)$$

A **solution** of the above system is a pair of differentiable functions $x = \phi_1(t)$, $y = \phi_2(t)$, defined on a common interval $I$, that satisfy each equation of the system on this interval.

# Initial-Value Problem

- On some interval $I$ containing $x_0$, the problem:

  Solve:

$$\frac{d^n y}{dx^n} = f(x, y, y', \ldots, y^{(n-1)})$$

  Subject to:

$$y(x_0) = y_0,\ y'(x_0) = y_1,\ \ldots\ ,\ y^{(n-1)}(x_0) = y_{n-1}$$

  where $y_0, y_1, \cdots, y_{n-1}$ are arbitrary real constants, is called an **initial-value problem (IVP).**

- The values of $y(x)$ and its first $n-1$ derivatives at a single point $x_0$: $y(x_0) = y_0$, $y'(x_0) = y_1$, ..., $y^{(n-1)}(x_0) = y_{n-1}$ are called the **initial conditions (ICs).**

# Initial-Value Problem

- Solving an $n$th-order IVP:

  Solve:

  $$\frac{d^n y}{dx^n} = f(x, y, y', \ldots, y^{(n-1)})$$

  Subject to:

  $$y(x_0) = y_0,\ y'(x_0) = y_1,\ \ldots\ ,\ y^{(n-1)}(x_0) = y_{n-1}$$

  entails first finding an $n$-parameter family of solutions of the given DE and then using the $n$ ICs at $x_0$ to determine numerical values of the $n$ constants in the family.

- The resulting particular solution is defined on some interval $I$ containing the initial point $x_0$.

# First-Order Initial-Value Problem

First-order initial-value problem:

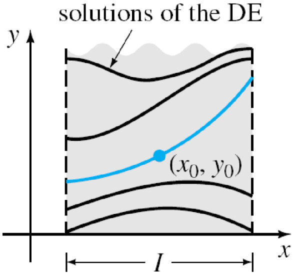
Solve:

$$\frac{dy}{dx} = f(x, y)$$

Subject to: $y(x_0) = y_0$

**Remark**

*We are seeking a solution $y(x)$ of the DE $y' = f(x, y)$ on an interval $I$ containing $x_0$ so that its graph passes through the specified point $(x_0, y_0)$.*



# Second-Order Initial-Value Problem

Second-order initial-value problem:

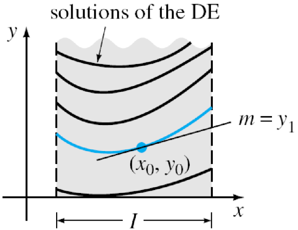
Solve:

$$\frac{d^2y}{dx^2} = f(x, y, y')$$

Subject to: $y(x_0) = y_0, \; y'(x_0) = y_1$

### Remark

*We are seeking a solution $y(x)$ of the DE $y'' = f(x, y, y')$ on an interval $I$ containing $x_0$ so that its graph not only passes through $(x_0, y_0)$ but the slope of the curve at this point is $y_1$.*



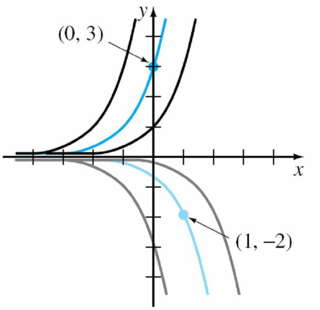
# Example 1: Two First-Order IVPs

The equation $y = ce^x$ is a one-parameter family of solutions of the simple 1st-order equation $y' = y$.

- The equation $y = 3e^x$ is a solution of the IVP: $y' = y, y(0) = 3$
- The equation $y = -2e^{x-1}$ is a solution of the IVP: $y' = y, y(1) = -2$

**Remark**

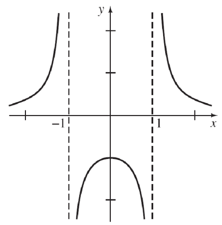
*Note that $y = 0$ is also a solution of the DE $y' = y$. ($c = 0$)*



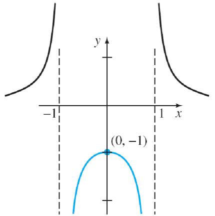
# Example 2: Interval $I$ of Definition of a Solution

A one-parameter family of solution of the 1st-order DE $y' + 2xy^2 = 0$ is $y = 1/(x^2 + c)$.
For a given IC: $y(0) = -1$, the solution is then $y = 1/(x^2 - 1)$. (Verify!)

- (a) shows the largest intervals on which $y = 1/(x^2 - 1)$ is a solution of the DE $y' + 2xy^2 = 0$ are $-\infty < x < -1$, $-1 < x < 1$, $1 < x < \infty$.
- (b) shows that the largest interval on which $y = 1/(x^2 - 1)$ is a solution of the IVP $y' + 2xy^2 = 0$, $y(0) = -1$ is $-1 < x < 1$.



**(a)** function defined for all $x$ except $x = \pm 1$



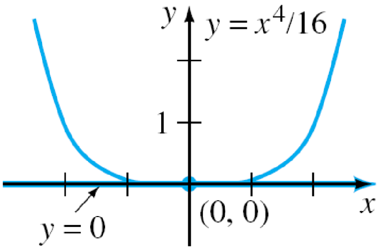
**(b)** solution defined on interval containing $x = 0$

# Example 4: An IVP Can Have Several Solutions

- Each of the functions $y = 0$ and $y = \frac{1}{16}x^4$ satisfies the DE $dy/dx = xy^{1/2}$ and IC $y(0) = 0$. (Verify this!)
- So the IVP

$$\frac{dy}{dx} = xy^{1/2}, \quad y(0) = 0$$

has at least two solutions.
- The graphs of both functions pass through the same point $(0, 0)$.



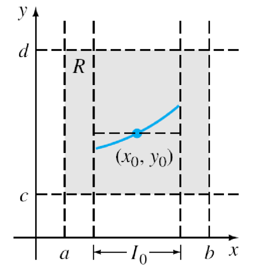
## Theorem (1.2.1: Existence of a Unique Solution)

*Let $R$ be a rectangular region in the $xy$-plane defined by $a \le x \le b$, $c \le y \le d$ that contains the point $(x_0, y_0)$ in its interior.*
*If $f(x,y)$ and $\partial f/\partial y$ are continuous on $R$, then there exits some interval $I_0 : x_0 - h < x < x_0 + h$, $h > 0$, contained in $a \le x \le b$, and a unique function $y(x)$, defined on $I_0$, that is a solution of the IVP:*

*Solve:*

$$\frac{dy}{dx} = f(x,y)$$

*Subject to:* $y(x_0) = y_0$



# Example 5: Example 4 Revisited

- The DE $dy/dx = xy^{1/2}$ possess at least two solutions whose graphs pass through $(0,0)$. (see Example 4)
- Inspection of the functions

$$f(x,y) = xy^{1/2} \qquad \text{and} \qquad \frac{\partial f}{\partial y} = \frac{x}{2y^{1/2}}$$

shows that they are continuous in the upper half-plane $y > 0$.

- Thus, Theorem 1.2.1 enables us to conclude that the DE has a unique solution for any IC given from the upper half-plane $y > 0$.

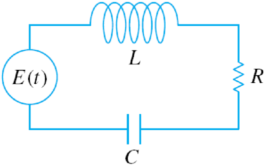
**Remark**

*Since the second equation is not continuous on $y = 0$, the solutions of the DE is not unique in the $xy$-plane! The conditions in Theorem 1.2.1 are sufficient but not necessary. (That is, the statement is not "iff".)*

# Differential Equations as Mathematical Models

- The mathematical description of a system or a phenomenon is called a **mathematical model**.
- Consider the single-loop series circuit shown in the figure, containing an inductor, resistor, and capacitor.
  According to **Kirchhoff's voltage law**, we can obtain a second-order differential equation describing the system

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = E(t)$$




**(a)** *LRC*-series circuit

# Homework

- Exercises 1.1: 9, 12, 17, 22, 36, 37.
- Exercises 1.2: 3, 8, 15, 20, 28.