Differential Equations

Lecture Set 02
First-Order Differential Equations

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Lineal Element

Let $y = y(x)$ be a solution of a 1st-order DE

$$\frac{dy}{dx} = f(x, y)$$

on its interval I of definition.

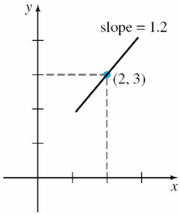
- The corresponding solution curve on I must have no breaks (i.e., continuous) and must possess a tangent line (i.e., differentiable) at each point $(x, y(x))$.
- The slope of the tangent line at $(x, y(x))$ on a solution curve is the value of the first derivative dy/dx at this point. (Due to the left-hand side of the DE.)
- The value $f(x, y)$ that the function f assigns to the point (x, y) represents the slope of a line, or a line segment called a **lineal element**. (Due to the right-hand side of the DE.)

Consider the equation

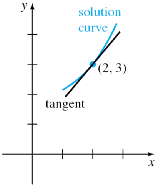
$$\frac{dy}{dx} = 0.2xy$$

where $f(x, y) = 0.2xy$.

At the point $(2, 3)$, the slope of a lineal element is $f(2, 3) = 1.2$ (Why?)



(a) lineal element at a point



(b) lineal element is tangent to solution curve that passes through the point

- (Note: Different (x, y) has a different lineal element.)

Direction Field

If we draw a *lineal element* at each point (x, y) of a rectangular grid on the xy-plane with slope $f(x, y)$, then the collection of these lineal elements is called a **direction field** of the differential equation

$$\frac{dy}{dx} = f(x, y)$$

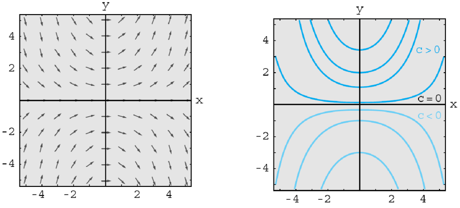
- The *direction field* suggests the appearance or shape of a family of solution curves of the DE. (Why?)
- *A single solution curve that passes through a direction field must follow the flow pattern of the field*; it is tangent to a lineal element when it intersects a point in the grid.

Example 1: Direction Field

Figure (a) shows the direction field for the differential equation

$$\frac{dy}{dx} = 0.2xy$$

Figure (b) shows *some* solution curves in the family $y = ce^{0.1x^2}$.



(a) direction field for $dy/dx = 0.2xy$

(b) some solution curves in the family $y = ce^{0.1x^2}$

- Remember that the value $0.2xy$ given by any (x, y) represents the slope at the point (x, y).

Example 2: Direction Field

Use a direction field to sketch an approximate solution curve for

$$\frac{dy}{dx} = \sin y, \quad y(0) = -\frac{3}{2}$$

Autonomous First-Order Differential Equations

- An ODE in which the *independent variable* does not appear *explicitly* is said to be **autonomous.**
- An autonomous 1st-order DE can be written as $F(y, y') = 0$ or in *normal form* as[1]

$$\frac{dy}{dx} = f(y)$$

Example

The 1st-order equations

$$\frac{dy}{dx} = 1 + y^2 \qquad \text{and} \qquad \frac{dy}{dx} = 0.2xy$$

are autonomous and nonautonomous, respectively.

[1] Note there is no "x" in the equation! To be more specific, we can write $y' = f(y)$.

Critical Points

Let

$$\frac{dy}{dx} = f(y) \tag{1}$$

be an autonomous 1st-order DE.

- A real constant c is a **critical point** of the autonomous DE if $f(c) = 0$.
- If c is a *critical point*, then $y(x) = c$ is a constant solution of the autonomous DE. (Why?)
- A constant solution $y(x) = c$ of Eq. (1) is called an **equilibrium solution**; equilibria are the *only* constant solutions of Eq. (1).

Example 3: An Autonomous DE (1/2)

The differential equation

$$\frac{dP}{dt} = P(a - bP)$$

where a and b are positive constants, has the *normal form*

$$\frac{dP}{dt} = f(P)$$

and hence is autonomous.[2]

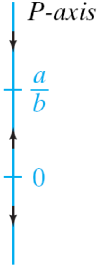
Since $f(P) = P(a - bP) = 0$,
$\Rightarrow$ 0 and $\frac{a}{b}$ are critical points of the equation (by definition),
$\Rightarrow$ the equilibrium solutions are $P(t) = 0$ and $P(t) = \frac{a}{b}$.

[2] (Note that P is a variable similar to the notation y used previously!)

The figure shown below is called a **phase portrait** of the DE

$$\frac{dP}{dt} = P(a - bP)$$

The vertical line is called a **phase line.**



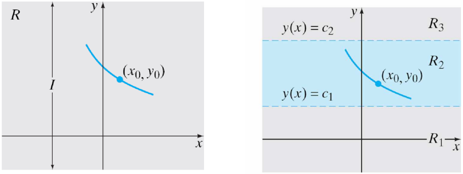
Interval	Sign of $f(P)$	$P(t)$	Arrow
$(-\infty, 0)$	minus	decreasing	points down
$(0, a/b)$	plus	increasing	points up
$(a/b, \infty)$	minus	decreasing	points down

- Note that $f(P)$ represents the slope here!
- (Verify the sign of $f(P)$.)

Solution Curves

Suppose $\frac{dy}{dx} = f(y)$ possesses two critical points c_1 and c_2, and $c_1 < c_2$.

- The graph of the equilibrium solutions $y(x) = c_1$ and $y(x) = c_2$ are horizontal lines partitioning region R into 3 subregions R_1, R_2, R_3.
- If (x_0, y_0) is in R_i, and $y(x)$ is a solution whose graph passes through this point, then $y(x)$ remains in the subregion R_i for all x.



(a) region R **(b)** subregions R_1, R_2, and R_3 of R

- $f(y)$ cannot change signs in a subregion. (Note the slope!)
- $y(x)$ is either increasing or decreasing in the subregion R_i.

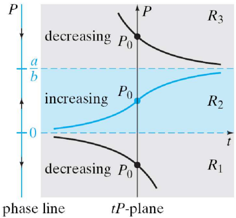
Example 4: Example 3 Revisited

For the differential equation

$$\frac{dP}{dt} = P(a - bP)$$

the three intervals determined on the P-axis or phase line by the critical points $P = 0$ and $P = \frac{a}{b}$ correspond in the tP-plane to three subregions:

$$R_1 : -\infty < P < 0, \quad R_2 : 0 < P < \frac{a}{b}, \quad \text{and} \quad R_3 : \frac{a}{b} < P < \infty$$



Separable Equation (1/2)

Definition (2.2.1: Separable Equation)

A first-order differential equation of the form

$$\frac{dy}{dx} = g(x)h(y) \tag{2}$$

is said to be **separable** or have **separable variables**.

Example

The equations

$$\frac{dy}{dx} = y^2 x e^{3x+4y} \qquad \text{and} \qquad \frac{dy}{dx} = y + \sin x$$

are separable and nonseparable, respectively.

Separable Equation (2/2)

Remark

Since (2) $\Rightarrow \dfrac{dy}{h(y)} = g(x)dx \Rightarrow \displaystyle\int \dfrac{dy}{h(y)} = \int g(x)dx.$

The solution of Eq. (2) is given by $\displaystyle\int \dfrac{1}{h(y)}dy = \int g(x)dx$

Example 1: Solving a Separable DE

Solve

$$(1 + x)dy - ydx = 0$$

Example 2: Solution Curve

Solve the initial-value problem

$$\frac{dy}{dx} = -\frac{x}{y}, \quad y(4) = -3$$

Example 3: Losing a Solution

Solve

$$\frac{dy}{dx} = y^2 - 4$$

Example 4: An Initial Value Problem

Solve

$$(e^{2y} - y)\cos x \frac{dy}{dx} = e^{y} \sin 2x, \quad y(0) = 0$$

Linear Equation

Definition (2.3.1: Linear Equation)

A first-order differential equation of the form

$$a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

is said to be a **linear equation** in the dependent variable y.

- When $g(x) = 0$, the linear equation is said to be **homogeneous**; otherwise, it is **nonhomogeneous.**
- The **standard form** of a linear equation is given by

$$\frac{dy}{dx} + P(x)y = f(x)$$

We seek a solution on an interval I for which both coefficient functions P and f are continuous.

First-Order Differential Equation (1/2)

The differential equation

$$\frac{dy}{dx} + P(x)y = f(x) \tag{3}$$

has the property that its **general solution** is the *sum* of the two solutions:

$$y = y_c + y_p$$

where y_c is a solution of the associated homogeneous equation

$$\frac{dy}{dx} + P(x)y = 0 \tag{4}$$

and y_p is a *particular solution* of the nonhomogeneous equation (3).

First-Order Differential Equation (2/2)

Since Eq. (4), i.e.

$$\frac{dy}{dx} + P(x)y = 0$$

is separable, y_c can be found by integrating the equation

$$\frac{dy}{y} + P(x)dx = 0$$

Thus,

$$y_c = ce^{-\int P(x)dx}$$

(Verify this! Note that c could be positive or negative.)

Note that Eq. (4) is a homogeneous differental equation!

Solving a Linear First-Order Equation:[3]

$$\frac{dy}{dx} + P(x)y = f(x)$$

1. Find the **integrating factor** $e^{\int P(x)dx}$.
2. Multiple the equation by the integrating factor.
   The left-hand side of the resulting equation is automatically the derivative of the integrating factor and y: [4]

$$\frac{d}{dx}[e^{\int P(x)dx}y] = e^{\int P(x)dx}f(x) \tag{5}$$

3. Integrate both sides of Eq. (5).

[3] Note this equation is nonhomogeneous, and different from the previous one!

[4] (Verify this!)

How to Derive the Integration Factor? (1/3)

To solve the differential equation

$$\frac{dy}{dx} + P(x)y = f(x) \tag{6}$$

we consider the form

$$\frac{d}{dx}[\mu(x)y] = \mu(x)f(x) \tag{7}$$

Why? Because y can be derived by integrating both sides of (7).
So now the problem is how to make (7) equivalent to (6):

$$(7) \Rightarrow \mu(x)\frac{dy}{dx} + y\frac{d\mu}{dx} = \mu(x)f(x) \Rightarrow \frac{dy}{dx} + \frac{1}{\mu(x)}\frac{d\mu}{dx}y = f(x)$$

Compare the last equation with (6), we need to have

$$\frac{1}{\mu(x)}\frac{d\mu}{dx} = P(x) \tag{8}$$

How to Derive the Integration Factor? (2/3)

From (8), we have

$$\frac{d\mu}{\mu} = P(x)dx$$

and it can be solved as[5]

$$\mu(x) = e^{\int P(x)dx} \tag{9}$$

The function $\mu(x)$ in (9) is actually the integration factor in Slide 22.

[5] Or more precisely, $\mu(x) = ce^{\int P(x)dx}$

How to Derive the Integration Factor? (3/3)

Since we force (7) equivalent to (6) and obtain (9), solving

$$\frac{dy}{dx} + P(x)y = f(x)$$

is equivalent to solving

$$\frac{d}{dx}[e^{\int P(x)dx}y] = e^{\int P(x)dx}f(x)$$

This concludes the derivation of the method using integrating factor.

Example 1: Solving a Homogeneous Linear DE

Solve

$$\frac{dy}{dx} - 3y = 0$$

Any better ways to solve this?

Remark

Integrating factor $\leftrightarrow$ nonhomogeneous linear DE

It is clearly not necessary to use the integrating factor to solve a homogeneous linear DE!

Example 2: Solving a Nonhomogeneous Linear DE

Solve

$$\frac{dy}{dx} - 3y = 6$$

How to verify $y = y_c + y_p$ from the previous example?

Example 3: General Solution

Solve

$$x\frac{dy}{dx} - 4y = x^6 e^x, \quad \text{for } x > 0$$

Example 4: General Solution

Find the general solution of

$$(x^2 - 9)\frac{dy}{dx} + xy = 0$$

(The DE is actually homogeneous and separable! another easy way...)

Example 5: An Initial-Value Problem

Solve

$$\frac{dy}{dx} + y = x, \quad y(0) = 4$$

Example 6: An Initial-Value Problem

Solve

$$\frac{dy}{dx} + y = f(x), \quad y(0) = 0 \quad \text{where} \quad f(x) = \begin{cases} 1, & 0 \leq x \leq 1 \\ 0, & x > 1 \end{cases}$$

Differential of A Function of Two Variables

If $z = f(x, y)$ is a function of two variables with continuous first partial derivatives in a region R of the xy-plane, then its differential is

$$dz = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

- In a special case when $f(x, y) = c$, where c is a constant, then we have

$$\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = 0 \tag{10}$$

- In other words, given a one-parameter family of functions $f(x, y) = c$, we can generate a first-order differential equation by computing the differential of both sides of the equality.

Example

If $x^2 - 5xy + y^3 = c$, then Eq. (10) gives the first-order DE

$$(2x - 5y)dx + (-5x + 3y^2)dy = 0$$

or

$$\frac{dy}{dx} = \frac{2x - 5y}{5x - 3y^2}$$

Exact Equation (1/2)

Definition (2.4.1: Exact Equation)

A differential expression

$$M(x,y)dx + N(x,y)dy$$

is an **exact differential** in a region R of the xy-plane if it corresponds to the differential of *some function* $f(x,y)$ defined in R.[a]

A first-order differential equation of the form

$$M(x,y)dx + N(x,y)dy = 0 \tag{11}$$

is said to be an **exact equation** if the expression on the left-hand side is an exact differential.

[a] Check the previous example, $x^2 - 5xy + y^3$ is an exact differential.

Exact Equation (2/2)

Remark

Eq. (11) can be derived from

$$M(x, y)dx + N(x, y)dy = df(x, y)$$

with $f(x, y) = c$, where c is a constant.

For example,

$$x^2y^3dx + x^3y^2dy = 0$$

is an exact equation since

$$d\left(\frac{1}{3}x^3y^3\right) = x^2y^3dx + x^3y^2dy$$

on the left-hand side of the equation.

Exact Differential

Theorem (2.4.1: Criterion for an Exact Differential)

Let $M(x,y)$ and $N(x,y)$ be continuous and have continuous first partial derivatives in a rectangular region R defined by $a < x < b$, $c < y < d$. Then a necessary and sufficient condition that

$$M(x,y)dx + N(x,y)dy$$

be an exact differential is

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

(Check the textbook for proof.)

Method of Solution

- Given an equation in the differential form

$$M(x,y)dx + N(x,y)dy = 0 \tag{12}$$

determine whether the equality

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

holds.

- If it does, then there exists a function f for which[6]

$$\frac{\partial f}{\partial x} = M(x,y) \qquad \text{or} \qquad \frac{\partial f}{\partial y} = N(x,y)$$

The solution of the given DE (12) is $f(x,y) = c$.

[6] Why? (Check the proof.)

Example 1: Solving an Exact DE

Solve

$$2xydx + (x^2 - 1)dy = 0$$

Example 2: Solving an Exact DE

Solve

$$(e^{2y} - y\cos xy)dx + (2xe^{2y} - x\cos xy + 2y)dy = 0$$

Example 3: An Initial-Value Problem

Solve

$$\frac{dy}{dx} = \frac{xy^2 - \cos x \sin x}{y(1 - x^2)}, \quad y(0) = 2$$

Substitutions (1/2)

Suppose we transform the first-order differential equation

$$\frac{dy}{dx} = f(x, y) \tag{13}$$

by the **substitution** $y = g(x, u)$, where u is regarded as a function of variable x, i.e. $u = u(x)$, then

$$\frac{dy}{dx} = \frac{\partial g}{\partial x}\frac{dx}{dx} + \frac{\partial g}{\partial u}\frac{du}{dx} \qquad \text{or} \qquad \frac{dy}{dx} = g_x(x, u) + g_u(x, u)\frac{du}{dx}$$

Thus, the first-order differential equation (13) becomes

$$g_x(x, u) + g_u(x, u)\frac{du}{dx} = f(x, g(x, u)) \qquad \text{or} \qquad \frac{du}{dx} = F(x, u)$$

which can be used to solve for $u = \phi(x)$. (Why?)

Substitutions (2/2)

Consequently, a solution of the original differential equation (13), i.e.

$$\frac{dy}{dx} = f(x, y)$$

is given by

$$y = g(x, \phi(x))$$

Remark

Note that Solutions by Substitutions *is a "two-step" process!*

Homogeneous Equations (1/2)

If a function f possesses the property

$$f(tx, ty) = t^{\alpha} f(x, y)$$

for some real number α, then f is said to be a **homogeneous function** of degree α.

Example

The function

$$f(x, y) = x^3 + y^3$$

is a homogeneous function of degree 3.

$$\begin{aligned} f(tx, ty) &= (tx)^3 + (ty)^3 = t^3x^3 + t^3y^3 \\ &= t^3(x^3 + y^3) = t^3 f(x, y) \end{aligned}$$

□

Homogeneous Equations (2/2)

A 1st-order DE in differential form

$$M(x,y)dx + N(x,y)dy = 0 \tag{14}$$

is said to be **homogeneous** if both the functions M and N are homogeneous of the *same* degree.

- In other words, Eq. (14) is homogeneous if

$$M(tx,ty) = t^{\alpha}M(x,y) \qquad \text{and} \qquad N(tx,ty) = t^{\alpha}N(x,y)$$

- Either of the substitutions $y = ux$ or $x = vy$, where u and v are new dependent variables, will reduce a homogeneous equation to a *separable* 1st-order DE.

Example 1: Solving a Homogeneous DE

Solve

$$(x^2 + y^2)dx + (x^2 - xy)dy = 0$$

The differential equation[7]

$$\frac{dy}{dx} + P(x)y = f(x)y^n \tag{15}$$

where n is any real number, is called a **Bernoulli's equation.**

Remark

For $n \neq 0$ and $n \neq 1$, the substitution $u = y^{1-n}$ reduces any equation of the form (15) to a linear equation.

[7]Note that Eq. (15) is a nonlinear differential equation.

Example 2: Solving a Bernoulli DE

Solve

$$x\frac{dy}{dx} + y = x^2y^2, \quad \text{for } x > 0$$

Reduction to Separation of Variables

A differential equation of the form

$$\frac{dy}{dx} = f(Ax + By + C)$$

can always be reduced to an equation with separable variables by means of the substitution $u = Ax + By + C$, where $B \neq 0$.

Example 3: An Initial-Value Problem

Solve

$$\frac{dy}{dx} = (-2x + y)^2 - 7, \quad y(0) = 0$$

A Numerical Method (1/2)

- Suppose the first-order initial-value problem

$$y' = f(x, y), \quad y(x_0) = y_0$$

has a solution.

- The **linearization** of the solution at $x = x_0$ is defined as

$$L(x) = y_0 + (x - x_0)f(x_0, y_0)$$

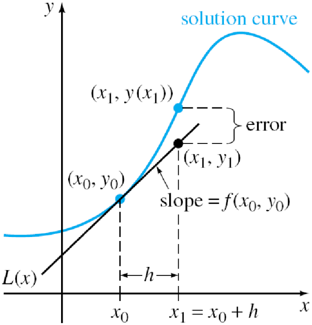
Its graph is a straight line tangent to the graph of $y = y(x)$ at point (x_0, y_0).
[See the next slide.]

A Numerical Method (2/2)

- Let h be a positive increment of the x-axis as shown below, then we have

$$y_1 = y_0 + hf(x_0, y_0)$$

where $y_1 = L(x_1)$.



$$L(x) = y_0 + (x - x_0)f(x_0, y_0)$$

Euler's Method (1/2)

- The point (x_1, y_1) on the tangent line is an approximation to the point $(x_1, y(x_1))$ on the solution curve of the differential equation.
- The accuracy of the approximation $y_1 \approx y(x_1)$ depends heavily on the size of the increment h. Usually, we must choose this **step size** to be "reasonably small".
- We can repeat the process using a second "tangent line" at (x_1, y_1) to obtain the point (x_2, y_2) :

$$y(x_2) = y(x_0 + 2h) = y(x_1 + h) \approx y_1 + hf(x_1, y_1)$$

- Continue in this manner, the values $y_1, y_2, y_3, \ldots$, can be defined recursively by the general formula

$$y_{n+1} = y_n + hf(x_n, y_n)$$

where $x_n = x_0 + nh, n = 0, 1, 2, \ldots$.

Euler's Method (2/2)

Remark

This procedure of using successive "tangent lines" is called **Euler's method** *for finding numerical solutions of differential equations.*

Example 1: Euler's Method

Consider the initial-value problem

$$y' = 0.1\sqrt{y} + 0.4x^2, \quad y(2) = 4$$

Use Euler's method to obtain an approximation of $y(2.5)$ using first $h = 0.1$ and then $h = 0.05$.

Remark

A very important aspect of numerical solutions is the *error analysis*. It will not be covered in this course. Students who are interested in this topic should refer to *Numerical Analysis* courses!

More sophisticated numerical methods for solving differential equations (such as the improved Euler method, RK4, etc.) will be covered in the future.

Homework

- Exercises 2.1: 2, 20, 23.
- Exercises 2.2: 5, 19, 28.
- Exercises 2.3: 8, 18, 29.
- Exercises 2.4: 3, 13, 21, 29.
- Exercises 2.5: 7, 12, 19, 25, 30.
- Exercises 2.6: 4.