Differential Equations

Lecture Set 04
Higher-Order Differential Equations

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Initial-Value Problem

For a linear differential equation, an ***n*th-order initial-value problem** is

Solve:

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

Subject to:

$$y(x_0) = y_0,\ y'(x_0) = y_1,\ \ldots\ ,\ y^{(n-1)}(x_0) = y_{n-1}$$

- For this problem we seek a function defined on some interval I, containing x_0, that satisfies the differential equation and the n initial conditions specified at x_0: $y(x_0) = y_0$, $y'(x_0) = y_1$, $\ldots$, $y^{(n-1)}(x_0) = y_{n-1}$.

Existence of a Unique Solution (1/2)

Theorem (4.1.1: Existence of a Unique Solution)

Let $a_n(x), a_{n-1}(x), \cdots, a_1(x), a_0(x)$ and $g(x)$ be continuous on an interval I, and let $a_n(x) \neq 0$ for every x in this interval. If $x = x_0$ is any point in this interval, then a solution $y(x)$ of the initial value problem

Solve:

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

Subject to:

$$y(x_0) = y_0,\ y'(x_0) = y_1,\ \ldots,\ y^{(n-1)}(x_0) = y_{n-1}$$

exists on the interval and is unique.

Existence of a Unique Solution (2/2)

Remark

The requirements in Theorem 4.1.1 that $a_i(x), i = 1, 2, \ldots, n$ be continuous and $a_n(x) \neq 0$ for every x in I are both important. Specifically, if $a_n(x) = 0$ for some x in the interval, then the solution of a linear IVP may not be unique or even exist.

Example

Due to Theorem 4.1.1, the initial-value problem

$$3y''' + 5y'' - y' + 7y = 0, \; y(1) = 0, \; y'(1) = 0, \; y''(1) = 0$$

has a unique solution $y = 0$ on any interval containing $x = 1$.

Example 2: Unique Solution of an IVP

The function

$$y = 3e^{2x} + e^{-2x} - 3x \quad (1)$$

is a solution of the initial-value problem

$$y'' - 4y = 12x,\ y(0) = 4,\ y'(0) = 1 \quad (2)$$

Since $a_2(x) = 1 \neq 0$ on any interval I containing $x = 0$, the given function (1) is a unique solution of (2) on the interval I.

Boundary-Value Problem

A problem such as

Solve:

$$a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

Subject to:

$$y(a) = y_0,\ y(b) = y_1$$

is called a **boundary-value problem (BVP)**.

The prescribed values $y(a) = y_0$ and $y(b) = y_1$ are called **bounrady conditions (BCs)**.

Boundary Conditions

For a 2nd-order DE, the pairs of boundary conditions could be

$$\begin{aligned} y(a) &= y_0, \quad y(b) = y_1 \\ y'(a) &= y_0, \quad y(b) = y_1 \\ y(a) &= y_0, \quad y'(b) = y_1 \\ y'(a) &= y_0, \quad y'(b) = y_1 \end{aligned}$$

where y_0 and y_1 denote arbitrary constants.

Remark

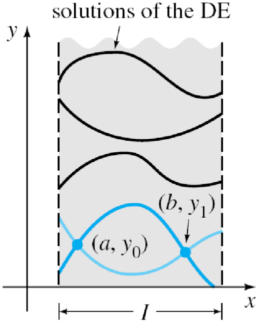
A BVP can have many, one, or no solutions.
(An example is shown in the next slide.)

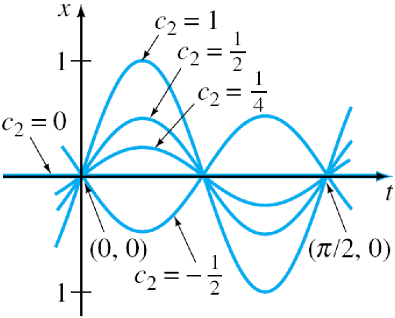
A BVP Can Have Many, One, or No Solutions

The two-parameter family of solutions of the differential equation $x'' + 16x = 0$ is $x = c_1 \cos 4t + c_2 \sin 4t$.

- If $x(0) = x(\frac{\pi}{2}) = 0$, then $x'' + 16x = 0$ has *infinitely many solutions*.
- If $x(0) = x(\frac{\pi}{8}) = 0$, the DE $x'' + 16x = 0$ as *a unique solution* $x = 0$.
- If $x(0) = 0,\ x(\frac{\pi}{2}) = 1$, the DE $x'' + 16x = 0$ has *no solution*.

(In the above cases, $x(0) = 0$ implies $c_1 = 0$.)





Homogeneous Equations

A linear nth-order differential equation of the form

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = 0 \tag{3}$$

is said to be **homogeneous**, whereas an equation

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x) \tag{4}$$

with $g(x)$ not identically zero, is said to be **nonhomogeneous.**

Remark

To solve a nonhomogeneous linear equation (4), we must first be able to solve the associated homogeneous equation (3).

Superposition Principle (1/2)

Theorem (4.1.2: Superposition Principle - Homogeneous Equations)

Let $y_1, y_2, \cdots, y_k$ be solutions of the homogeneous nth-order DE

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = 0$$

on an interval I. Then the linear combination

$$y = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_k y_k(x)$$

where the c_i, $i = 1, 2, \cdots, k$ are arbitrary constants, is also a solution on the interval.

Superposition Principle (2/2)

Corollary

A constant multiple $y = c_1 y_1(x)$ of a solution $y_1(x)$ of a homogeneous linear DE is also a solution.
Thus, a homogeneous linear DE always possesses the trivial solution $y = 0$.

Example (4: Superposition – Homogeneous DE)

The function $y_1 = x^2$ and $y_2 = x^2 \ln x$ are both solutions of the homogeneous linear equation $x^3 y''' - 2xy' + 4y = 0$ on the interval $(0, \infty)$. (Verify.)

Thus, $y = c_1 x^2 + c_2 x^2 \ln x$ is also a solution of the equation on the interval $(0, \infty)$.

Linear Dependence/Independence (1/2)

Definition (4.1.1: Linear Dependence/Independence)

A set of functions $f_1(x), f_2(x), \cdots, f_n(x)$ is said to be **linearly dependent** on an interval I if there exist constants, $c_1, c_2, \cdots, c_n$, not all zero, such that

$$c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) = 0$$

for *every* x in the interval.
If the set of functions is *not* linearly dependent on the interval, it is said to be **linearly independent**.

Linear Dependence/Independence (2/2)

Remark

In other words, a set of functions is linearly independent on an interval I *if the only constants for which*

$$c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) = 0$$

for every x *in the interval are* $c_1 = c_2 = \cdots = c_n = 0$.

Example 5: Linear Dependent Set of Functions

The set of functions

$$f_1(x) = \cos^2 x, \quad f_2(x) = \sin^2 x, \quad f_3(x) = \sec^2 x, \quad f_4(x) = \tan^2 x$$

is linearly dependent on the interval $(-\pi/2, \pi/2)$ since

$$1 \cdot \cos^2 x + 1 \cdot \sin^2 x + (-1) \cdot \sec^2 x + 1 \cdot \tan^2 x = 0$$

Here we have $c_1 = 1, c_2 = 1, c_3 = -1, c_4 = 1$.

Example 6: Linear Dependent Set of Functions

The set of functions

$$f_1(x) = \sqrt{x} + 5, \quad f_2(x) = \sqrt{x} + 5x, \quad f_3(x) = x - 1, \quad f_4(x) = x^2$$

is linearly dependent on the interval $(0, \infty)$ since

$$f_2(x) = 1 \cdot f_1(x) + 5 \cdot f_3(x) + 0 \cdot f_4(x)$$

Here we have $c_1 = 1, c_2 = -1, c_3 = 5, c_4 = 0$.

Wronskian

Theorem (4.1.2: Wronskian)

Suppose each of the functions $f_1(x), f_2(x), \cdots, f_n(x)$ *possesses at least* $n-1$ *derivatives. The determinant*

$$W(f_1, f_2, \cdots, f_n) = \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & \vdots & & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix}$$

is called the **Wronskian** *of the functions.*

Criterion for Linearly Independent Solutions

Theorem (4.1.3: Criterion for Linearly Independent Solutions)

Let $y_1, y_2, \cdots, y_n$ be solutions of the homogeneous nth-order DE

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = 0$$

on an interval I.

Then the set of solutions is **linearly independent** *on I if and only if*

$$W(y_1, y_2, \cdots, y_n) \neq 0$$

for every x in the interval.

Fundamental Set of Solutions (1/2)

Definition (4.1.3: Fundamental Set of Solutions)

Any set $y_1, y_2, \cdots, y_n$ of n linearly independent solutions of the homogeneous linear nth-order DE

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = 0$$

on an interval I is said to be a **fundamental set of solution** on the interval.

Fundamental Set of Solutions (2/2)

Theorem (4.1.4: Existence of a Fundamental Set)

There exists a fundamental set of solutions for the homogeneous linear nth-order DE

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = 0$$

on an interval I.

General Solution – Homogeneous Equations

Theorem (4.1.5: General Solution – Homogeneous Equations)

Let $y_1, y_2, \cdots, y_n$ be a fundamental set of solutions of the homogeneous linear nth-order DE

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = 0$$

on an interval I. Then the **general solution** *of the equation on the interval is*

$$y = c_1y_1(x) + c_2y_2(x) + \cdots + c_ny_n(x)$$

where the c_i, $i = 1, 2, \cdots, n$ are arbitrary constants.

Example 7: General Solution of a Homogeneous DE

The functions $y_1 = e^{3x}$ and $y_2 = e^{-3x}$ are both solutions of the homogeneous linear equation $y'' - 9y = 0$ on the interval $(-\infty, \infty)$.

$$(e^{3x})'' - 9e^{3x} = 9e^{3x} - 9e^{3x} = 0$$

and

$$(e^{-3x})'' - 9(e^{-3x}) = 9(e^{-3x}) - 9(e^{-3x}) = 0$$

Since the Wronskian

$$W(e^{3x}, e^{-3x}) = \begin{vmatrix} e^{3x} & e^{-3x} \\ 3e^{3x} & -3e^{-3x} \end{vmatrix} = -6 \neq 0$$

for every x, the functions y_1 and y_2 form a fundamental set of solutions and $y = c_1 e^{3x} + c_2 e^{-3x}$ is the general solution of the equation on the interval.

Example 7: General Solution of a Homogeneous DE

The function $y = 4\sinh 3x - 5e^{3x}$ is a solution of the above DE

$$y'' - 9y = 0$$

since

$$\left(4 \cdot \frac{e^{3x} - e^{-3x}}{2} - 5e^{3x}\right)'' - 9 \cdot \left(4\frac{e^{3x} - e^{-3x}}{2} - 5e^{3x}\right) = 0$$

This solution can be written as the form of $y = c_1 e^{3x} + c_2 e^{-3x}$ with $c_1 = 2$ and $c_2 = -7$.
It is indeed a linear combination of a fundamental set of solutions e^{3x}, e^{-3x}.

Example 8: A Solution Obtained from a General Sol.

The functions $y_1 = e^x,\ y_2 = e^{2x},\ y_3 = e^{3x}$ satisfy the 3rd-order equation

$$y''' - 6y'' + 11y' - 6y = 0$$

Since

$$W(e^x, e^{2x}, e^{3x}) = \begin{vmatrix} e^x & e^{2x} & e^{3x} \\ e^x & 2e^{2x} & 3e^{3x} \\ e^x & 4e^{2x} & 9e^{3x} \end{vmatrix} = 2e^{6x} \neq 0$$

for every x, the functions y_1, y_2, and y_3 form a fundamental set of solutions on $(-\infty, \infty)$.

Thus, $y = c_1e^x + c_2e^{2x} + c_3e^{3x}$ is the general solution of the DE on the interval.

General Solution – Nonhomogeneous Equations

Theorem (4.1.6: General Solution – Nonhomogeneous Equations)

Let y_p be any **particular solution** *of the nonhomogeneous linear nth-order differential equation (4) on an interval I, and let $y_1, y_2, \cdots, y_n$ be a fundamental set of solutions of the associated homogeneous differential equation (3) on I.*
Then the **general solution** *of the equation on the interval is*

$$y = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x) + y_p$$

where the c_i, $i = 1, 2, \cdots, n$ are arbitrary constants.

Complementary Function

The general solution of a nonhomogeneous linear equation consists of the sum of two functions:

$$y = y_c(x) + y_p(x)$$

The linear combination $y_c(x) = c_1y_1(x) + c_2y_2(x) + \cdots + c_ny_n(x)$, which is the general solution of the homogeneus DE, is called the **complementary function** for the nonhomogeneous DE.

Remark

To solve a nonhomogeneous linear DE, we first solve the associated homogeneous equation and then find any particular solution of the nonhomogeneous equation.
The general solution of the nonhomogeneous equation is then

$$y = complementary\ function + any\ particular\ solution = y_c + y_p$$

Example 10: General Sol. of a Nonhomogeneous DE

It can be shown that the function

$$y_p = -\frac{11}{12} - \frac{1}{2}x$$

is a *particular solution* of the nonhomogeneous equation

$$y''' - 6y'' + 11y' - 6y = 3x \tag{5}$$

and

$$y_c = c_1 e^x + c_2 e^{2x} + c_3 e^{3x}$$

is the *general solution* of the associated homogeneous equation.

Thus, the general solution of (5) is given by

$$y = y_c + y_p = c_1 e^x + c_2 e^{2x} + c_3 e^{3x} - \frac{11}{12} - \frac{1}{2}x$$

Superposition Principle – Nonhomogeneous Eqs

Theorem (4.1.7: Superposition Principle – Nonhomogeneous Eqs)

Let $y_{p_1}, y_{p_2}, \cdots, y_{p_k}$ be particular solutions of the nonhomogeneous linear nth-order DE (4) on an interval I corresponding to k distinct functions $g_1, g_2, \cdots, g_k$. That is, suppose y_{p_i} denotes a particular solution of the corresponding DE

$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g_i(x)$$

where $i = 1, 2, \cdots, k$.
Then $y_p = y_{p_1}(x) + y_{p_2}(x) + \cdots + y_{p_k}(x)$ is a particular solution of

$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g_1(x) + g_2(x) + \cdots + g_k(x)$$

Example 11: Nonhomogeneous DE

It is shown that

$y_{p_1} = -4x^2$ is a particular solution of $y'' - 3y' + 4y = -16x^2 + 24x - 8$
$y_{p_2} = e^{2x}$ is a particular solution of $y'' - 3y' + 4y = 2e^{2x}$
$y_{p_3} = xe^x$ is a particular solution of $y'' - 3y' + 4y = 4xe^x - e^x$

Thus,

$$y = y_{p_1} + y_{p_2} + y_{p_3} = -4x^2 + e^{2x} + xe^x$$

is a solution of

$$y'' - 3y' + 4y = -16x^2 + 24x - 8 + 2e^{2x} + 4xe^x - e^x$$

Remarks

A dynamic system whose mathematical model is a linear nth-order DE

$$a_n(t)y^{(n)} + a_{n-1}(t)y^{(n-1)} + \cdots + a_1(t)y' + a_0(t)y = g(t)$$

is said to be an nth-order **linear system.**

- The n time-dependent functions $y(t), y'(t), \ldots, y^{(n-1)}(t)$ are the **state variables** of the system. Their values at some time t give the **state of the system.**
- The function $g(t)$ is called the **input function**, **forcing function**, or **excitation function.**
- A solution $y(t)$ of the DE is said to be the **output** or **response of the system.**

Under the conditions stated in Theorem 4.1.1, the response $y(t)$ is uniquely detemined by the input and the state of the system prescribed at a time t_0 – i.e., by the initial conditions $y(t_0), y'(t_0), \ldots, y^{(n-1)}(t_0)$.

Reduction of Order

Suppose that y_1 denotes a nontrivial solution of

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$$

and that y_1 is defined on an interval I. *We seek a second solution, y_2, so that y_1, y_2 is a linearly independent set on the interval I.*

Is It Possible? How to Achieve This?

- If y_1 and y_2 are linearly independent, then y_2/y_1 is *non*-constant.
- That is, $y_2(x)/y_1(x) = u(x)$ or $y_2(x) = u(x)y_1(x)$.
- The function $u(x)$ can be found by substituting

  $$y_2(x) = u(x)y_1(x)$$

  into the given DE.
- This method is called **reduction of order** because we must solve a linear 1st-order DE to find u.

Example 1: A Second Solution by Reduction of Order

Given that $y_1 = e^x$ is a solution of

$$y'' - y = 0$$

on the interval $(-\infty, \infty)$, use *reduction of order* to find a second solution y_2.

Example 2: A Second Solution

The function $y_1 = x^2$ is a solution of

$$x^2y'' - 3xy' + 4y = 0$$

Find the general solution of the DE on the interval $(0, \infty)$.

Auxiliary Equation

Consider the second-order differential equation

$$ay'' + by' + cy = 0 \tag{6}$$

where a, b, and c are constants. The associated **auxiliary equation** of the DE is defined as

$$am^2 + bm + c = 0 \tag{7}$$

Let m_1 and m_2 be two roots of Eq. (7), then the solutions of Eq. (6) are given by

$y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$	if m_1 and m_2 are real and distinct
$y = c_1 e^{m_1 x} + c_2 x e^{m_1 x}$	if m_1 and m_2 are real and equal
$y = e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x)$	if m_1 and m_2 are complex conjugates

Formula Derivation (1/2)

If we try to find a solution of the form $y = e^{mx}$, then Eq. (6) becomes

$$e^{mx}(am^2 + bm + c) = 0 \tag{8}$$

Since $e^{mx} \neq 0$ for all x, we have the auxiliary equation (7) hold.

Case I: Distinct Real Roots If Eq. (7) has two distinct real roots, then we find two solutions $y_1 = e^{m_1 x}$ and $y_2 = e^{m_2 x}$, which are linearly independent. Thus, the general solution of Eq. (6) is given by

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$$

Case II: Repeated Real Roots If $m_1 = m_2$, one solution is $y_1 = e^{m_1 x}$, and the second solution can be obtained by *reduction of order* described in the previous section as $y_2 = x e^{m_1 x}$. Thus, the general solution of Eq. (6) is given by

$$y = c_1 e^{m_1 x} + c_2 x e^{m_1 x}$$

Case III: Conjugate Complex Roots If m_1 and m_2 are complex conjugates, let $m_1 = \alpha + i\beta$ and $m_2 = \alpha - i\beta$. Since they are distinct, the general solution is

$$y = c_1 e^{(\alpha+i\beta)x} + c_2 e^{(\alpha-i\beta)x} \tag{9}$$

From the Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$, we have $e^{i\beta x} = \cos\beta x + i\sin\beta x$ and $e^{-i\beta x} = \cos\beta x - i\sin\beta x$, which implies $e^{i\beta x} + e^{-i\beta x} = 2\cos\beta x$ and $e^{i\beta x} - e^{-i\beta x} = 2i\sin\beta x$. If we choose $c_1 = c_2 = 1$ and $c_1 = 1, c_2 = -1$ in Eq. (9), then two linearly independent solutions are given by

$$y_1 = e^{(\alpha+i\beta)x} + e^{(\alpha-i\beta)x} = 2e^{\alpha x}\cos\beta x$$

and

$$y_2 = e^{(\alpha-i\beta)x} - e^{(\alpha-i\beta)x} = 2ie^{\alpha x}\sin\beta x$$

Thus, the general solution is given by

$$y = c_1 e^{\alpha x}\cos\beta x + c_2 e^{\alpha x}\sin\beta x = e^{\alpha x}(c_1\cos\beta x + c_2\sin\beta x)$$

Example 1: Second-Order DEs

Solve the following differential equations.

$(a)\ 2y'' - 5y' - 3y = 0 \qquad (b)\ y'' - 10y' + 25y = 0 \qquad (c)\ y'' + 4y' + 7y = 0$

Example 2: An Initial-Value Problem

Solve

$$4y'' + 4y' + 17y = 0, \quad y(0) = -1,\ y'(0) = 2$$

Higher-Order Equations

To solve an nth-order differential equation

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_2 y'' + a_1 y' + a_0 y = 0$$

where the $a_i, i = 0, 1, 2, \ldots, n$ are real constants, we must solve an nth-order polynomial equation

$$a_n m^n + a_{n-1} m^{n-1} + \cdots + a_2 m^2 + a_1 m + a_0 = 0$$

When m_1 is a root of multiplicity k of an nth-order auxiliary equation, then the linearly independent solutions are

$$e^{m_1 x},\ xe^{m_1 x},\ x^2 e^{m_1 x}, \cdots, x^{k-1} e^{m_1 x}$$

Example 3: Third-Order DE

Solve

$$y''' + 3y'' - 4y = 0$$

Example 4: Fourth-Order DE

Solve

$$\frac{d^4y}{dx^4} + 2\frac{d^2y}{dx^2} + y = 0$$

Method of Undetermined Coefficients (1/2)

The **method of undetermined coefficients** for solving a *nonhomogeneous* linear DE

$$a_n y^n + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = g(x) \tag{10}$$

is to *guess* the form of y_p according to the function $g(x)$.

The general method is limited to linear DEs such as Eq. (10) where

- the coefficients $a_i, i = 0, 1, \ldots, n$ are constants and
- $g(x)$ is a *constant*, a *polynomial*, an *exponential function*, a *sine or cosine function*, or *finite sums and products of these functions*.

Method of Undetermined Coefficients (2/2)

The set of functions that consists of constants, polynomials, exponentials, sines, and cosines has the property that *derivatives of their sums and products are again sums and products of constants, polynomials, exponentials, sines, and cosines*.

Because the linear combination of derivatives

$$a_n y_p^n + a_{n-1} y_p^{(n-1)} + \cdots + a_1 y_p' + a_0 y_p$$

must be identical to $g(x)$, it is reasonable to assume that y_p *has the same form as* $g(x)$.

Example 1: General Sol. Using Undetermined Coeffs

Solve

$$y'' + 4y' - 2y = 2x^2 - 3x + 6$$

Example 2: Particular Solution Using Undetermined Coefficients

Find a particular solution of

$$y'' - y' + y = 2\sin 3x$$

Example 3: Forming y_p by Superposition

Solve

$$y'' - 2y' - 3y = 4x - 5 + 6xe^{2x}$$

Example 4: A Glitch in the Method

Find a particular solution of

$$y'' - 5y' + 4y = 8e^{x}$$

Trial Particular Solutions

Forcing Term $f(t)$	Trial Forced Solution $x_f(t)$
k	A
t	$At + B$
t^2	$At^2 + Bt + C$
t^n	$At^n + Bt^{n-1} + \cdots + Yt + Z$
$\sin \omega t,\ \cos \omega t$	$A \sin \omega t + B \cos \omega t$
$e^{\sigma t} \sin \omega t,\ e^{\sigma t} \cos \omega t$	$e^{\sigma t}(A \sin \omega t + B \cos \omega t)$
$te^{\sigma t} \sin \omega t,\ te^{\sigma t} \cos \omega t$	$te^{\sigma t}(A \sin \omega t + B \cos \omega t)$ $+e^{\sigma t}(C \sin \omega t + D \cos \omega t)$

Trial Particular Solutions

Case I: No function in the assumed particular solution is a solution of the associated homogeneous differential equation.

Form Rule for Case I: The form of y_p is a linear combination of all linearly independent functions that are generated by *repeated differentiations* of $g(x)$.

Case II: A function in the assumed particular solution is also a solution of the associated homogeneous differential equation.

Form Rule for Case II: If any y_p contains terms that duplicate terms in y_c, then that y_p must be multiplied by x^n, where n is the smallest positive integer that eliminates that duplication.

Example 5: Forms of Particular Solutions – Case I

Determine the form of a particular solution of

$$(a)\ y'' - 8y' + 25y = 5x^3 e^{-x} - 7e^{-x} \qquad (b)\ y'' + 4y = x\cos x$$

Example 6: Forming y_p by Superposition – Case I

Determine the form of a particular solution of

$$y'' - 9y' + 14y = 3x^2 - 5\sin 2x + 7xe^{6x}$$

Example 7: Particular Solution – Case II

Find a particular solution of

$$y'' - 2y' + y = e^x$$

Example 8: An Initial-Value Problem

Solve

$$y'' + y = 4x + 10\sin x, \quad y(\pi) = 0, y'(\pi) = 2$$

Example 10: Third-Order DE – Case I

Solve

$$y''' + y'' = e^x \cos x$$

Example 11: Fourth-Order DE – Case II

Determine the form of a particular solution of

$$y^{(4)} + y''' = 1 - x^2 e^{-x}$$

Variation of Parameters (1/2)

How to Find a Particular Solution?

For the linear second-order differential equation

$$y'' + P(x)y' + Q(x)y = f(x) \tag{11}$$

we seek a particular solution of the form

$$y_p = u_1(x)y_1(x) + u_2(x)y_2(x)$$

where y_1 and y_2 form a fundamental set of solutions on I of the associated homogeneous form of

$$a_2(x)y'' + a_1(x)y' + a_0(x) = g(x)$$

The functions u_1 and u_2 are given by solving the system

$$\begin{cases} y_1u_1' + y_2u_2' = 0 \\ y_1'u_1' + y_2'u_2' = f(x) \end{cases} \tag{12}$$

Variation of Parameters (2/2)

The solutions u_1' and u_2' can be expressed in terms of determinants:

$$u_1' = \frac{W_1}{W} = -\frac{y_2 f(x)}{W} \quad \text{and} \quad u_2' = \frac{W_2}{W} = \frac{y_1 f(x)}{W} \tag{13}$$

where

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}, \quad W_1 = \begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix}, \quad W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix} \tag{14}$$

- The linear system (12) with two unknowns u_1' and u_2' can be solved by Cramer's rule.
- The function u_1 and u_2 are found by integrating the results in (13).
- The determinant W is the Wronskian of y_1 and y_2.
- By linear independence of y_1 and y_2 on I, we know that $W(y_1(x), y_2(x)) \neq 0$ for every x in the interval.

Derivation of Eq. (12)

Let $y = u_1 y_1 + u_2 y_2$, then

$$y' = u_1 y_1' + u_1' y_1 + u_2' y_2 + u_2 y_2'$$

and

$$y'' = u_1 y_1'' + u_1' y_1' + u_1'' y_1 + u_1' y_1' + u_2'' y_2 + u_2' y_2' + u_2' y_2 + u_2 y_2''$$

Thus, the left hand side of Eq. (11) becomes

$$\begin{aligned} y'' + Py' + Qy &= u_1[y_1'' + Py_1' + Qy_1] + u_2[y_2'' + Py_2' + Qy_2] \\ &\quad + y_1 u_1'' + u_1' y_1' + y_2 u_2'' + u_2' y_2' \\ &\quad + P[u_1' y_1 + u_2' y_2] + u_1' y_1' + u_2' y_2' \\ &= \frac{d}{dx}[u_1' y_1 + u_2' y_2] + P[u_1' y_1 + u_2' y_2] + u_1' y_1' + u_2' y_2' \end{aligned}$$

If we further let $u_1' y_1 + u_2' y_2 = 0$, then Eq. (11) becomes[1]

$$u_1' y_1' + u_2' y_2' = f(x)$$

[1] An additional constraint posed on u_1 and u_2.

Summary of The Method (Variation of Parameters)

Procedure of solving the differential equation

$$a_2 y'' + a_1 y' + a_0 y = g(x)$$

using **variation of parameters:**

1. Find the complementary function $y_c = c_1 y_1 + c_2 y_2$.
2. Compute the Wronskian $W(y_1(x), y_2(x))$.
3. Put the equation into the standard form (dividing by a_2)

$$y'' + Py' + Qy = f(x)$$

4. Find u_1 and u_2 by integrating $u_1' = W_1/W$ and $u_2' = W_2/W$, where W_1 and W_2 are defined as in Eq. (14). [Or, solve Eq. (12).]
5. A particular solution is $y_p = u_1 y_1 + u_2 y_2$.
6. The general solution of the equation is then $y = y_c + y_p$.

Example 1: General Solution Using VOP

Solve

$$y'' - 4y' + 4y = (x+1)e^{2x}$$

Example 2: General Solution Using VOP

Solve

$$4y'' + 36y = \csc 3x$$

Higher-Order Equations

Given a linear nth-order differential equation in the standard form

$$y^{(n)} + P_{n-1}(x)y^{(n-1)} + \cdots + P_1(x)y' + P_0(x)y = f(x)$$

If

$$y_c = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n$$

is the complementary function, then a particular solution is

$$y_p = u_1(x)y_1(x) + u_2(x)y_2(x) + \cdots + u_n(x)y_n(x)$$

where $u_k', k = 1, 2, \cdots, n$ are determined by

$$u_k' = \frac{W_k}{W}$$

where W is the Wronskian of $y_1, y_2, \cdots, y_n$ and W_k is the determinant obtained by replacing the kth column of the Wronskian by the column consisting of the right-hand side of the equations.

Remarks

- **Variation of parameters** has a distinct advantage over the **method of undetermined coefficients** in that it will *always* yield a particular solution y_p provided that the associated homogeneous equation can be solved.
- Variation of parameters, unlike undetermined coefficients, is applicable to linear DE with *variable coefficients*.[2]

[2] Check Example 5 in Section 4.7.

Cauchy-Euler Equation

A linear differential equation of the form

$$a_n x^n \frac{d^n y}{dx^n} + a_{n-1} x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1 x \frac{dy}{dx} + a_0 y = g(x)$$

where the coefficients $a_n, a_{n-1}, \cdots, a_0$ are constants, is known as a **Cauchy-Euler equation.**

2nd-Order Homogeneous Cauchy-Euler Equation (1)

For the second-order homogeneous Cauchy-Euler equation

$$ax^2\frac{d^2y}{dx^2} + bx\frac{dy}{dx} + cy = 0 \tag{15}$$

If we substitute $y = x^m$, where m is to be determined, the 2nd-order equation becomes[3]

$$(am(m-1) + bm + c)x^m = 0$$

Thus, $y = x^m$ is a solution of the DE whenever m is a solution of the **auxiliary equation**

$$am(m-1) + bm + c = 0$$

[3] Check!

2nd-Order Homogeneous Cauchy-Euler Equation (2)

There are three different cases to be considered, depending on whether the roots of this quadratic equation are *real and distinct*, *real and equal*, or *complex*.

Case I: Distinct Real Roots. Let m_1 and m_2 denote the real roots of $am(m-1)+bm+c=0$ such that $m_1 \neq m_2$. Then the general solution of Eq. (15) is

$$y = c_1 x^{m_1} + c_2 x^{m_2}$$

Case II: Repeated Real Roots. Let m denotes the repeated real root of the characteristic equation. Then the general solution of Eq. (15) is

$$y = c_1 x^m + c_2 x^m \ln x$$

2nd-Order Homogeneous Cauchy-Euler Equation (3)

Case III: Conjugate Complex Roots. If the roots of the characteristic equation are the conjugate pair $\alpha \pm i\beta$, where α and $\beta > 0$ are real. Then the general solution of Eq. (15) is

$$y = x^{\alpha}[c_1 \cos(\beta \ln x) + c_2 \sin(\beta \ln x)]$$

Case I: Distinct Real Roots

Since $y = x^{m_1}$ and $y = x^{m_2}$ are both solutions of Eq. (15) and $m_1 \neq m_2$, they form a fundamental set of solutions. Thus, the general solution is

$$y = c_1 x^{m_1} + c_2 x^{m_2}$$

Case II: Repeated Real Roots

The repeated real roots of the auxiliary equation is $m = \dfrac{a-b}{2a}$.

Since one solution is $y_1 = x^m$, the other solution can then be obtained from **reduction of order**.

Case III: Conjugate Complex Roots

If the conjugate complex roots of the auxiliary equation are $\alpha \pm \beta i$, then a solution of Eq. (15) is

$$y = c_1 x^{\alpha+\beta i} + c_2 x^{\alpha-\beta i} = x^{\alpha}(c_1 x^{\beta i} + c_2 x^{-\beta i})$$

By Euler's formula,

$$x^{i\beta} = (e^{\ln x})^{i\beta} = e^{i\beta \ln x} = \cos(\beta \ln x) + i \sin(\beta \ln x) \quad (16)$$

$$x^{-i\beta} = \cos(\beta \ln x) - i \sin(\beta \ln x) \quad (17)$$

Let $c_1 = 1, c_2 = 1$ and $c_1 = -i, c_2 = i$, we have $2\cos(\beta \ln x)$ and $2\sin(\beta \ln x)$ in the complex number terms (16) and (17).[4]

That is, we can choose $\cos(\beta \ln x)$ and $\sin(\beta \ln x)$ as two independent solutions (their Wronskian is not zero).[5]

Thus, the general solution is

$$y = x^{\alpha}[c_1 \cos(\beta \ln x) + c_2 \sin(\beta \ln x)]$$

---

[4] To get real solutions!

[5] This is something like *change of basis* in linear algebra.

Example 1: Distinct Roots

Solve

$$x^2 \frac{d^2y}{dx^2} - 2x\frac{dy}{dx} - 4y = 0$$

Example 2: Repeated Roots

Solve

$$4x^2\frac{d^2y}{dx^2}+8x\frac{dy}{dx}+y=0$$

Example 3: An Initial-Value Problem

Solve

$$4x^2y'' + 17y = 0, \quad y(1) = -1, y'(1) = -\frac{1}{2}$$

Example 4: Third-Order Equation

Solve

$$x^3 \frac{d^3y}{dx^3} + 5x^2 \frac{d^2y}{dx^2} + 7x \frac{dy}{dx} + 8y = 0$$

Example 5: Variation of Parameters

Solve

$$x^2y'' - 3xy' + 3y = 2x^4e^x$$

Initial-Value Problem of Nonhomogeneous DE

The solution $y(t)$ of the second order initial-value problem

$$y'' + P(x)y' + Q(x)y = f(x), \quad y(x_0) = y_0, \quad y'(x_0) = y_1 \tag{18}$$

can be expressed as the superposition of two solutions:

$$y(x) = y_h(x) + y_p(x)$$

where $y_h(x)$ is the solution of the associated *homogeneous DE* with *nonhomogeneous initial conditions*

$$y'' + P(x)y' + Q(x)y = 0, \quad y(x_0) = y_0, \quad y'(x_0) = y_1 \tag{19}$$

and $y_p(x)$ is the solution of the *nonhomogeneous DE* with *homogeneous initial condisitions*

$$y'' + P(x)y' + Q(x)y = f(x), \quad y(x_0) = 0, \quad y'(x_0) = 0. \tag{20}$$

Remarks

- If the coefficients $P(x)$ and $Q(x)$ are constants, then the IVP (19), i.e.,
$$y'' + P(x)y' + Q(x)y = 0, \quad y(x_0) = y_0, \quad y'(x_0) = y_1$$
can be solved by the method described in Section 4.3.
- For the IVP (20), i.e.,
$$y'' + P(x)y' + Q(x)y = f(x), \quad y(x_0) = 0, \quad y'(x_0) = 0$$
the zero initial conditions are given. It can be thought as the description of a physical system which is initially at rest. Thus, the solution of the IVP (20) is sometimes called a *rest solution*.
- That is, the solution of the IVP of nonhomogeneous DE (18) is given by the solutions of "zero forcing term (19)" + "zero initial condition (20)".

Green's Function (1/2)

If $y_1(x)$ and $y_2(x)$ forms the fundamental set of solutions of the DE

$$y'' + P(x)y' + Q(x)y = 0 \tag{21}$$

then a particular solution of the nonhomogeneous DE

$$y'' + P(x)y' + Q(x)y = f(x) \tag{22}$$

can be found by variation of parameters (Section 4.6), i.e.,

$$y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x) \tag{23}$$

where

$$u_1'(x) = -\frac{y_2(x)f(x)}{W(x)}, \quad u_2'(x) = \frac{y_1(x)f(x)}{W(x)}, \quad W(x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix}$$

on an interval I where all functions make sense.

Green's Function (2/2)

If x and x_0 are numbers in I, then (23) can be written as

$$\begin{aligned} y_p(x) &= y_1(x)\int_{x_0}^{x}\frac{-y_2(t)f(t)}{W(t)}dt + y_2(x)\int_{x_0}^{x}\frac{y_1(t)f(t)}{W(t)}dt \\ &= \int_{x_0}^{x}\frac{-y_1(x)y_2(t)}{W(t)}f(t)dt + \int_{x_0}^{x}\frac{y_1(t)y_2(x)}{W(t)}f(t)dt \end{aligned}$$

where $W(t) = W(y_1(t), y_2(t))$ is the Wronskian.

The above equation can be rewritten as

$$y_p(x) = \int_{x_0}^{x} G(x,t)f(t)dt \tag{24}$$

where

$$G(x,t) = \frac{y_1(t)y_2(x) - y_1(x)y_2(t)}{W(t)} \tag{25}$$

is called the **Green's function** for the differential equation (18).

Remarks

- Observe that a Green's function (25), i.e.,

$$G(x,t) = \frac{y_1(t)y_2(x) - y_1(x)y_2(t)}{W(t)}$$

  depends only on the fundamental solutions $y_1(x)$ and $y_2(x)$ of the associated homogeneous DE for (22) and *not* on the forcing function $f(x)$.
- Thus, all linear second-order differential equations (22) with the same left-hand side but with different forcing functions have the same Green's function.
- Eq. (25) can also be called as the *Green's function for the second-order differential operator* $L = D^2 + P(x)D + Q(x)$.

Example 1: Particular Solution

Use the Green's function to find a particular solution of

$$y'' - y = f(x)$$

Theorem (Theorem 4.8.1: Solution of the IVP (20))

The function

$$y_p(x) = \int_{x_0}^{x} G(x,t)f(t)dt \tag{26}$$

is the solution of the initial-value problem

$$y'' + P(x)y' + Q(x)y = f(x), \quad y(x_0) = 0, \quad y'(x_0) = 0.$$

(Note the zero initial conditions!)

(Check the textbook for proof, page 172. It requires Leibniz formula.)

Example 3

Solve the initial-value problem

$$y'' - y = 1/x, \quad y(1) = 0, \quad y'(1) = 0.$$

Example 4

Solve the initial-value problem

$$y'' + 4y = x, \quad y(0) = 0, \quad y'(0) = 0.$$

Solution of the IVP (18)

Theorem (Theorem 4.8.2: Solution of the IVP (18))

If $y_h(x)$ is the solution of the initial-value problem

$$y'' + P(x)y' + Q(x)y = 0, \quad y(x_0) = y_0, \quad y'(x_0) = y_1$$

and $y_p(x)$ is the solution (26) of the initial-value problem

$$y'' + P(x)y' + Q(x)y = f(x), \quad y(x_0) = 0, \quad y'(x_0) = 0$$

on the interval I, then

$$y(x) = y_h(x) + y_p(x)$$

is the solution of the initial-value problem

$$y'' + P(x)y' + Q(x)y = f(x), \quad y(x_0) = y_0, \quad y'(x_0) = y_1$$

Proof of Theorem 4.8.2

Check the DE:
Since $y_h(x)$ is a linear combination of the fundamental solutions,
$\Rightarrow y = y_h + y_p$ is a solution of the nonhomogeneous DE.

Check the IC:
Since y_h satisfies the initial conditions in (19) and y_p satisfies the initial condition in (20), we have

$$y(x_0) = y_h(x_0) + y_p(x_0) = y_0 + 0 = y_0$$
$$y'(x_0) = y_h'(x_0) + y_p'(x_0) = y_1 + 0 = y_1$$

Thus, both the differential equation and initial conditions are satisfied. □

Remark

Eq. (19) has a forcing function, and Eq. (20) does not have any forcing function.

According to Theorem 4.8.2, the response $y(x)$ of a physical system described by the IVP (18) can be separated into two different responses, $y_h(x)$ and $y_p(x)$. That is,

$$y(x) = y_h(x) + y_p(x)$$

where

- $y_h(x)$: response of system due to IC, $y(x_0) = y_0, y'(x_0) = y_1$.
- $y_p(x)$: response of system due to the forcing function $f(x)$.

Example 5

Solve the initial-value problem

$$y'' + 4y = \sin 2x, \quad y(0) = 1, \quad y'(0) = -2. \tag{27}$$

Homework

- Exercises 4.1: 4, 9, 26, 34.
- Exercises 4.2: 3, 13, 20.
- Exercises 4.3: 5, 22, 35, 41.
- Exercises 4.4: 5, 13, 30, 38.
- Exercises 4.6: 4, 14, 21, 26.
- Exercises 4.7: 7, 22, 36, 38.
- Exercises 4.8: 3, 13, 22.