Differential Equations

Lecture Set 06
Series Solutions of Linear Equations

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Power Series

A power series in $x - a$ is an infinite series of the form

$$\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$$

Such a series is also said to be a **power series centered at a.**

Convergence:

- A power series

$$\sum_{n=0}^{\infty} c_n(x-a)^n$$

  is convegent at a specified value of x if its sequence of *partial sum* $\{S_N(x)\}$ converges.[1]
- That is, $\lim_{N\to\infty} S_N(x) = \lim_{N\to\infty} \sum_{n=0}^{N} c_n(x-a)^n$ exists.
- If the limit does not exist at x, then the series is said to be *divergent*.

[1] The partial sum $S_N(x) = \sum_{n=0}^{N} c_n(x-a)^n$.

Properties of Power Series (2/5)

Interval of Convergence:

- Every power series has an *interval of convergence*.
- The interval of convergence is the set of all real numbers x for which the series converges.

Radius of Convergence:

- Every power series has a *radius of convergence* R. If $R > 0$, then the power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ converges for $|x-a| < R$ and diverges for $|x-a| > R$.
- A series might or might not converge at the endpoints $a - R$ and $a + R$ of this interval.
- If the series converges only at its center a, then $R = 0$. If the series converges for all x, then we write $R = \infty$.

Properties of Power Series (3/5)

Absolute Convergence:

- If x is a number in the interval of convergence and is not an endpoint of the interval, then the series of absolute value $\sum_{n=0}^{\infty} |c_n(x-a)^n|$ converges.

Ratio Test:

- Test "convergence" of a power series can often be determined by the *ratio test*: Suppose that $c_n \neq 0$ for all n and that

$$\lim_{n\to\infty} \left| \frac{c_{n+1}(x-a)^{n+1}}{c_n(x-a)^n} \right| = |x-a| \lim_{n\to\infty} \left| \frac{c_{n+1}}{c_n} \right| = L$$

- If $L < 1$, the series converges absolutely; If $L > 1$, the series diverges; and if $L = 1$, the test is inconclusive.

Example

The interval of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{(x-3)^n}{2^n n}$$

is $[1, 5)$. The radius of convergence is 2.

A Power Series Defines a Function:

- A power series defines a function

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$$

  whose domain is the interval of convergence of the series.
- If the radius of convergence is $R > 0$, then f is continuous, differentiable, and integrable on the interval $(a-R, a+R)$.
- Moreover, $f'(x)$ and $\int f(x)dx$ can be found by *term-by-term* differentiation and integration.

Properties of Power Series (5/5)

Identity Property:

- If $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n = 0$, $R > 0$ for all numbers x in the interval of convergence, then $c_n = 0$ for all n.

Analytic at a Point:

- A function f is *analytic* at a point a if it can be represented by a power series in $x - a$ with a positive or infinite radius of convergence.
- For example, e^x, $\sin x$, $\cos x$ are analytic at $x = 0$ by Taylor series expansion.

Arithmetic of Power Series:

- Power series can be combined through the operations of addition, multiplication, and division.

Example 1: Adding Two Power Series

Write

$$\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + \sum_{n=0}^{\infty} c_n x^{n+1} \tag{1}$$

as a single power series whose general terms involves x^k.

Ordinary and Singular Points

Definition (6.2.1: Ordinary and Singular Points)

A point x_0 is said to be an **ordinary point** of the differential equation

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$$

if both $P(x)$ and $Q(x)$ in the standard form

$$y'' + P(x)y' + Q(x)y = 0$$

are analytic at x_0. A point that is not an ordinary point is said to be a **singular point** of the equation.

Example

Verify that every finite value of x is an ordinary point of the DE

$$y'' + (e^x)y' + (\sin x)y = 0$$

Polynomial Coefficients

Given a 2nd order linear differential equation

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0 \tag{2}$$

and its standard form

$$y'' + P(x)y' + Q(x)y = 0 \tag{3}$$

- A *polynomial* is analytic at any value x, and a *rational function* is analytic *except* at points where its denominator is zero.
- Thus, if $a_2(x)$, $a_1(x)$, and $a_0(x)$ in Eq. (2) are polynomials with no common factors, then both rational functions $P(x)$ and $Q(x)$ in Eq. (3) are analytic except where $a_2(x) = 0$.
- It follows that x_0 is an ordinary point if $a_2(x_0) \neq 0$ whereas $x = x_0$ is a singular point if $a_2(x_0) = 0$.

Example

Find the singular points of

$$(x^2 - 1)y'' + 2xy' + 6y = 0 \tag{4}$$

and

$$(x^2 + 1)y'' + xy' - y = 0 \tag{5}$$

Existence of Power Series Solutions (1/2)

Theorem (6.2.1: Existence of Power Series Solutions)

If $x = x_0$ is an ordinary point of the differential equation

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$$

we can always find two linearly independent solutions in the form of a power series centered at x_0. That is, $y = \sum_{n=0}^{\infty} c_n(x - x_0)^n$. A series solution converges at least on some interval defined by $|x - x_0| < R$, where R is the distance from x_0 to the closest singular point.

Existence of Power Series Solutions (2/2)

Remark

A solution of the form

$$y = \sum_{n=0}^{\infty} c_n (x - x_0)^n$$

is said to be a **solution about the ordinary point** x_0. *The distance* R *in Theorem 6.2.1 is the* minimum value *or the* lower bound *for the radius of convergence of series solutions of the differential equation about* x_0.

Example 2: Lower Bound for Radius of Convergence

The complex number $1 \pm 2i$ are singular points[2] of the differential equation

$$(x^2 - 2x + 5)y'' + xy' - y = 0$$

Since $x = 0$ is an ordinary point of the equation, we can find *two* linearly independent power series solutions about 0.

The solutions have the form

$$y = \sum_{n=0}^{\infty} c_n x^n$$

and will converge *at least* for $|x| < \sqrt{5}$, because the radius of convergence is $\sqrt{5}$. $(0 \leftrightarrow 1 \pm 2i : \sqrt{5})$

In fact, one of the solutions is valid on $(-\infty, \infty)$.

[2] Verify.

Example 3: Power Series Solutions

Solve

$$y'' + xy = 0 \tag{6}$$

Example 5: Three-Terms Recurrence Relation

Solve

$$y'' - (1 + x)y = 0$$

Example 6: DE with Nonpolynomial Coefficients

Solve

$$y'' + (\cos x)y = 0$$

Regular and Irregular Singular Point (1/2)

Definition (6.2: Regular Singular Point)

A singular point x_0 is said to be a **regular singular point** of the DE

$$y'' + P(x)y' + Q(x)y = 0 \tag{7}$$

if the functions $p(x) = (x - x_0)P(x)$ and $q(x) = (x - x_0)^2 Q(x)$ are both analytic at x_0.

A singular point that is *not* regular is said to be an **irregular singular point** of the equation.

Recap

The point x_0 is an ordinary point of (7) if $P(x)$ and $Q(x)$ are both analytic at x_0. The point x_0 is a singular point of (7) if either $P(x)$ or $Q(x)$ is not analytic at x_0.

Regular and Irregular Singular Point (2/2)

Remark

If $x - x_0$ *appears* at most *to the first power in the denominator of* $P(x)$ *and* at most *to the second power in the denominator of* $Q(x)$*, then* $x = x_0$ *is a regular singular point.*
Moreover, the original DE can be put into the form

$$(x - x_0)^2 y'' + (x - x_0)p(x)y' + q(x)y = 0 \tag{8}$$

where p *and* q *are analytic at* $x = x_0$*.*

Note that (8) can be written as

$$y'' + \frac{p(x)}{(x - x_0)}y' + \frac{q(x)}{(x - x_0)^2}y = 0$$

in the standard form.

Example 1: Classification of Singular Points

Classify the singular points of the DE

$$(x^2 - 4)^2 y'' + 3(x - 2)y' + 5y = 0$$

Frobenius' Theorem

Theorem (6.2: Frobenius' Theorem)

If $x = x_0$ is a regular singular point of the differential equation

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$$

then there exists at least one *solution of the form*

$$y = (x - x_0)^r \sum_{n=0}^{\infty} c_n (x - x_0)^n = \sum_{n=0}^{\infty} c_n (x - x_0)^{n+r}$$

where the number r is a constant to be determined. The series will converge at least on some interval $0 < x - x_0 < R$.

Example 2: Two Series Solutions

Solve

$$3xy'' + y' - y = 0$$

Bessel's Equation

The differential equation

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0 \tag{9}$$

is called **Bessel's equation of order** ν.

Bessel Functions of The First Kind

The solutions of Bessel's equation are usually denoted by

$$J_\nu(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(1+\nu+n)} \left(\frac{x}{2}\right)^{2n+\nu}$$

and

$$J_{-\nu}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(1-\nu+n)} \left(\frac{x}{2}\right)^{2n-\nu}$$

where $\Gamma(\alpha)$ is the gamma function with the property $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$.

The functions $J_\nu(x)$ and $J_{-\nu}(x)$ are called **Bessel's functions of the first kind** of order ν and $-\nu$, respectively.

The general solution of Bessel's equation on $(0, \infty)$ is

$$y = c_1 J_\nu(x) + c_2 J_{-\nu}(x), \quad \nu \neq \text{integer}$$

Bessel Functions of The Second Kind

If $\nu \neq$ integer, the function defined by the linear combination

$$Y_\nu(x) = \frac{\cos \nu\pi J_\nu(x) - J_{-\nu}(x)}{\sin \nu\pi}$$

and the function $J_\nu(x)$ are linearly independent solution of Bessel's equation (9).
Thus another form of the general solution of (9) is

$$y = c_1 J_\nu(x) + c_2 Y_\nu(x)$$

provided $\nu \neq$ integer.

In fact, *any* value of ν the general solution of (9) on $(0, \infty)$ can be written as

$$y = c_1 J_\nu(x) + c_2 Y_\nu(x)$$

$Y_\nu(x)$ is called the **Bessel function of the second kind** of order ν.

Legendre's Equation

The differential equation

$$(1 - x^2)y'' - 2xy' + n(n+1)y = 0$$

is called **Legendre's equation of order n.**

Legendre Polynomials

The polynomials

$$
\begin{aligned}
P_0(x) &= 1, & P_1(x) &= x \\
P_2(x) &= \frac{1}{2}(3x^2 - 1), & P_3(x) &= \frac{1}{2}(5x^3 - 3x) \\
P_4(x) &= \frac{1}{8}(35x^4 - 30x^2 + 3), & P_5(x) &= \frac{1}{8}(63x^5 - 70x^3 + 15x)
\end{aligned}
$$

are called **Legendre polynomials** and denoted by $P_n(x)$. They are the solutions of Legendre's equation.

(Give it a try!)

Legendre Polynomials

Legendre polynomials can be derived from the recurrence relation

$$(k+1)P_{k+1}(x) - (2k+1)xP_k(x) + kP_{k-1}(x) = 0, \qquad k = 1, 2, 3, \ldots$$

or generated by differentiation

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n}(x^2 - 1)^n, \qquad n = 0, 1, 2, \ldots$$

Homework

- Exercises 6.1: 2, 9, 22, 29.
- Exercises 6.2: 4, 15, 22.
- Exercises 6.3: 4, 13.