Differential Equations

Lecture Set 07
The Laplace Transform

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Integral Transform (1/2)

A definite integral such as

$$\int_a^b K(s,t)f(t)dt$$

transforms a function f of the variable t into a function F of the variable s.

For an **integral transform**, the interval of integration is the *unbounded* interval $[0,\infty)$. □

Integral Transform (2/2)

If $f(t)$ is defined for $t \geq 0$, then the *improper integral* $\int_0^\infty K(s,t)f(t)dt$ is defined as a limit:

$$\int_0^\infty K(s,t)f(t)dt = \lim_{b\to\infty} \int_0^b K(s,t)f(t)dt \qquad (1)$$

If the limit exists, then the integral exists or is **convergent**; if the limit does not exist, the integral does not exist and is **divergent**. □

The function $K(s,t)$ in Eq. (1) is called the **kernel** of the transform. The choice

$$K(s,t) = e^{-st}$$

as the kernel gives us an especially important integral transform. □

Laplace Transform

Definition (7.1.1: Laplace Transform)

Let f be a function defined for $t \geq 0$. Then the integral

$$\mathscr{L}\{f(t)\} = \int_0^{\infty} e^{-st} f(t)\,dt$$

is said to be the **Laplace transform** of f, provided that the integral converges.

Example 1: Applying Definition 7.1.1

Evaluate $\mathscr{L}\{1\}$.

Example 2: Applying Definition 7.1.1

Evaluate $\mathscr{L}\{t\}$.

Example 3: Applying Definition 7.1.1

Evaluate $\mathscr{L}\left\{e^{-3t}\right\}$.

Example 4: Applying Definition 7.1.1

Evaluate $\mathscr{L}\{\sin 2t\}$.

$\mathscr{L}$ Is a Linear Transform

For a linear combination of functions we can write

$$\int_0^\infty e^{-st}[\alpha f(t) + \beta g(t)]dt = \alpha \int_0^\infty e^{-st} f(t)dt + \beta \int_0^\infty e^{-st} g(t)dt$$

whenever both integrals converge for $s > c$.

Hence it follows that

$$\mathscr{L}\{\alpha f(t) + \beta g(t)\} = \alpha \mathscr{L}\{f(t)\} + \beta \mathscr{L}\{g(t)\} = \alpha F(s) + \beta G(s)$$

Because of the linearity property, $\mathscr{L}$ is said to be a **linear transform.**

Example

$$\mathscr{L}\{1 + 5t\} = \mathscr{L}\{1\} + 5\mathscr{L}\{t\} = \frac{1}{s} + \frac{5}{s^2}$$

Transforms of Some Basic Functions

Theorem (7.1.1: Transforms of Some Basic Functions)

$$\mathscr{L}\{\delta(t)\} = 1$$

$$\mathscr{L}\{1\} = \frac{1}{s}$$

$$\mathscr{L}\{t^n\} = \frac{n!}{s^{n+1}}, \quad n = 1, 2, 3, \cdots$$

$$\mathscr{L}\left\{e^{at}\right\} = \frac{1}{s-a}$$

$$\mathscr{L}\left\{te^{at}\right\} = \frac{1}{(s-a)^2}$$

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

$$\mathscr{L}\{\sin kt\} = \frac{k}{s^2 + k^2}$$

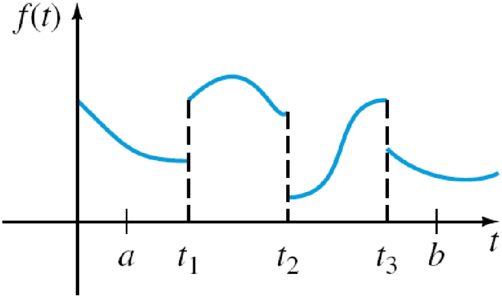
$$\mathscr{L}\{\cos kt\} = \frac{s}{s^2 + k^2}$$

$$\mathscr{L}\{\sinh kt\} = \frac{k}{s^2 - k^2}$$

$$\mathscr{L}\{\cosh kt\} = \frac{s}{s^2 - k^2}$$

Piecewise Continuous

A function f is **piecewise continuous** on $[0, \infty)$ if, in any interval $0 \leq a \leq t \leq b$, there are at most a *finite* number of points t_k, $k = 1, 2, \cdots, n$ at which f has *finite* discontinuities and is continuous on each open interval $t_{k-1} < t < t_k$.



Exponential Order

Definition (7.1.2: Exponential Order)

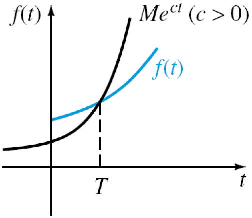
A function f is said to be of **exponential order** c if there exist constants c, $M > 0$, and $T > 0$ such that

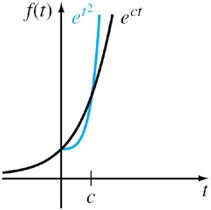
$$|f(t)| \leq Me^{ct} \quad \text{for all} \quad t > T$$

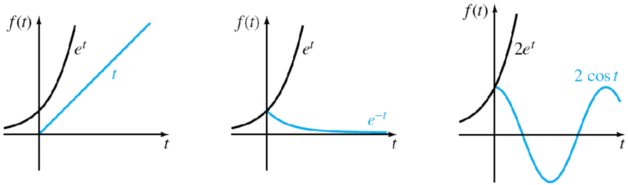
Remark

If f is an increasing *function, then the condition $|f(t)| \leq Me^{ct}$, $t > T$, simply states that the graph of f on the interval (T, ∞) does not grow faster than the graph of the exponential function Me^{ct}, where c is a positive constant.*

Example







(a) **(b)** **(c)**

Sufficient Conditions for Existence

Theorem (7.1.2: Sufficient Conditions for Existence)

If f is piecewise continuous on $[0, \infty)$ and of exponential order c, then $\mathscr{L}\{f(t)\}$ exists for $s > c$.

Proof.

$$\mathscr{L}\{f(t)\} = \underbrace{\int_0^T e^{-st} f(t)dt}_{\text{exists}} + \underbrace{\int_T^\infty e^{-st} f(t)dt}_{\text{exists?}} \tag{2}$$

Since f is of exponential order, $\Rightarrow \exists\ c, M > 0, T > 0$ s.t. $|f(t)| \leq Me^{ct}$ for $t > T$

$|\int_T^\infty e^{-st} f(t)dt| \leq \int_T^\infty |e^{-st}||f(t)|dt \leq M\int_T^\infty e^{-(s-c)t}dt = M\frac{e^{-(s-c)T}}{s-c}$ for $s > c$.

$\Rightarrow \int_T^\infty e^{-st} f(t)dt$ is bounded and exists for $s > c$.

Thus, $\mathscr{L}\{f(t)\}$ exists by Eq. (2). □

Example 5: Transform of a Piecewise Continuous Function

Evaluate $\mathscr{L}\{f(t)\}$ where

$$f(t) = \begin{cases} 0, & 0 \le t < 3 \\ 2, & t \ge 3 \end{cases}$$

Behavior of $F(s)$ as $s \to \infty$

Theorem (7.1.3: Behavior of $F(s)$ as $s \to \infty$)

If f is piecewise continuous on $(0, \infty)$ and of exponential order and $F(s) = \mathscr{L}\{f(t)\}$, then $\lim\limits_{s\to\infty} F(s) = 0$.

Proof.

f is of exponential order c, $\Rightarrow \exists\ \gamma, M_1 > 0, T > 0$ s.t. $|f(t)| \le M_1 e^{\gamma t}$ for $t > T$.

f is piecewise continuous on $[0, T]$,

$\Rightarrow f$ is bounded on $[0, T] \Rightarrow |f(t)| \le M_2 = M_2 e^{0t}$.

Let $M = \max\{M_1, M_2\}$ and $c = \max\{\gamma, 0\}$, then

$$|F(s)| \le \int_0^\infty e^{-st}|f(t)|dt \le M \int_0^\infty e^{-(s-c)t}dt = \tfrac{M}{s-c} \text{ for } s > c.$$

$\Rightarrow \lim_{s\to\infty} |F(s)| = 0.$

$\Rightarrow \lim_{s\to\infty} F(s) = 0.$ $\square$

The Inverse Problem

If $F(s)$ represents the Laplace transform of a function $f(t)$, that is, $\mathscr{L}\{f(t)\} = F(s)$, we then say $f(t)$ is the **inverse Laplace transform** of $F(s)$ and write

$$f(t) = \mathscr{L}^{-1}\{F(s)\}$$

Remark

In the application of the Laplace transform to equations we are not able to determine an unknown function $f(t)$ directly; rather, we are able to solve for the Laplace transform $F(s)$ of $f(t)$; but from that knowledge we ascertain f by computing $f(t) = \mathscr{L}^{-1}\{F(s)\}$.

Some Inverse Transforms

Theorem (7.2.1: Some Inverse Transforms)

$$1 = \mathscr{L}^{-1}\left\{\frac{1}{s}\right\}$$

$$t^n = \mathscr{L}^{-1}\left\{\frac{n!}{s^{n+1}}\right\}$$

$$e^{at} = \mathscr{L}^{-1}\left\{\frac{1}{s-a}\right\}$$

$$\sin kt = \mathscr{L}^{-1}\left\{\frac{k}{s^2+k^2}\right\}$$

$$\cos kt = \mathscr{L}^{-1}\left\{\frac{s}{s^2+k^2}\right\}$$

$$\sinh kt = \mathscr{L}^{-1}\left\{\frac{k}{s^2-k^2}\right\}$$

$$\cosh kt = \mathscr{L}^{-1}\left\{\frac{s}{s^2-k^2}\right\}$$

Example 1: Applying Theorem 7.2.1

Evaluate (a) $\mathscr{L}^{-1}\left\{\frac{1}{s^5}\right\}$ (b) $\mathscr{L}^{-1}\left\{\frac{1}{s^2+7}\right\}$.

Example 2: Termwise Division and Linearity

Evaluate $\mathscr{L}^{-1}\left\{\dfrac{-2s+6}{s^2+4}\right\}$.

Example 3: Partial Fractions: Distinct Linear Factors

Evaluate $\mathscr{L}^{-1}\left\{\dfrac{s^2+6s+9}{(s-1)(s-2)(s+4)}\right\}.$

Rational Function

A **rational function** of s is the ratio of two *polynomials*

$$F(s) = \frac{a_m s^m + a_{m-1} s^{m-1} + \cdots + a_0}{b_n s^n + b_{n-1} s^{n-1} + \cdots + b_0} = \frac{N(s)}{D(s)}$$

The rational function is *proper* if $n > m$.

The objective is to break $F(s)$ into *simple* rational functions.

- The first step is to factor $D(s)$ into its elementary factors

$$D(s) = b_n (s - p_1)(s - p_2) \cdots (s - p_n)$$

  If all b_i's are real, then each p_k is either real or appears in complex conjugate pairs.

Partial Fraction Expansion: Case I

For the rational function

$$F(s) = \frac{a_m s^m + a_{m-1}s^{m-1} + \cdots + a_0}{b_n s^n + b_{n-1}s^{n-1} + \cdots + b_0} = \frac{N(s)}{D(s)}$$

with $n > m$, and all poles are *simple*, then

$$F(s) = \frac{A_1}{s - p_1} + \frac{A_2}{s - p_2} + \cdots + \frac{A_n}{s - p_n}$$

where

$$A_k = (s - p_k)F(s)|_{s=p_1}$$

The inverse Laplace transform is then

$$f(t) = A_1 e^{p_1 t} + A_2 e^{p_2 t} + \cdots + A_n e^{p_n t}$$

Partial Fraction Expansion: Case II (1/2)

For the rational function

$$\begin{aligned} F(s) &= \frac{a_m s^m + a_{m-1} s^{m-1} + \cdots + a_0}{b_n s^n + b_{n-1} s^{n-1} + \cdots + b_0} \\ &= \frac{a_m s^m + a_{m-1} s^{m-1} + \cdots + a_0}{b_n (s-p_1)^{n_1} (s-p_2)^{n_2} \cdots (s-p_k)^{n_k}} \end{aligned}$$

with $n > m$, and we allow *multiple poles*, then

$$F(s) = \frac{A_{11}}{(s-p_1)^{n_1}} + \frac{A_{12}}{(s-p_1)^{n_1-1}} + \cdots + \frac{A_{1n}}{s-p_1} + \text{other fractions}$$

Partial Fraction Expansion: Case II (2/2)

where

$$A_{11} = (s - p_1)^{n_1} F(s)|_{s=p_1}, \quad A_{12} = \frac{d}{ds}(s - p_1)^{n_1} F(s)|_{s=p_1},$$

$$A_{13} = \frac{1}{2}\frac{d^2}{ds^2}(s - p_1)^{n_1} F(s)|_{s=p_1}, \quad \cdots$$

$$A_{1n_1} = \frac{1}{n_1 - 1}\frac{d^{n_1-1}}{ds^{n_1-1}}(s - p_1)^{n_1} F(s)|_{s=p_1}$$

The inverse Laplace transform is then given by

$$f(t) = A_{11}\frac{t^{n_1-1}}{(n_1-1)!}e^{p_1 t} + A_{12}\frac{t^{n_1-2}}{(n_1-2)!}e^{p_1 t} + \cdots$$

Transform of a Derivative

Theorem (7.2.2: Transform of a Derivative)

If $f, f', \ldots, f^{(n-1)}$ are continuous on $[0, \infty)$ and are of exponential order and if $f^{(n)}(t)$ is piecewise continuous on $[0, \infty)$, then

$$\mathscr{L}\{f^{(n)}(t)\} = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - f^{(n-1)}(0)$$

where $F(s) = \mathscr{L}\{f(t)\}$. □

For example,

$$\mathscr{L}\{f'(t)\} = sF(s) - f(0)$$

Similarly,

$$\mathscr{L}\{f''(t)\} = s^2 F(s) - sf(0) - f'(0)$$
$$\mathscr{L}\{f'''(t)\} = s^3 F(s) - s^2 f(0) - sf'(0) - f''(0)$$

Example

Verify that

$$\mathscr{L}\{f'(t)\} = sF(s) - f(0)$$
$$\mathscr{L}\{f''(t)\} = s^2F(s) - sf(0) - f'(0)$$

Solving Linear ODEs (1/2)

From the result given in Theorem 7.2.2, $\mathscr{L}\{d^n y/dx^n\}$ depends on $Y(s) = \mathscr{L}\{y(t)\}$ and the $n-1$ derivatives of $y(t)$ evaluated at $t = 0$.

This property makes Laplace transform ideally suited for solving linear IVP in which the DE has *constant coefficients*.

Such a DE is simply a linear combination of terms $y, y', y'', \ldots, y^{(n)}$:

$$a_n \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_0 y = g(t)$$
$$y(0) = y_0, y'(0) = y_1, \ldots, y^{(n-1)}(0) = y_{n-1}$$

where the $a_i, i = 1, 2, \ldots, n$ and $y_0, y_1, \ldots, y_{n-1}$ are constants.

Solving Linear ODEs (2/2)

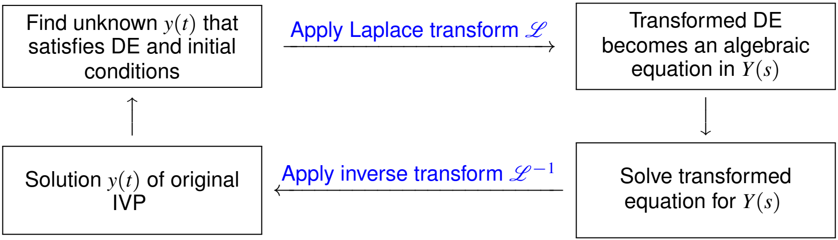
By the linearity property of Laplace transform and Theorem 7.2.2, we have

$$a_n[s^n Y(s) - s^{n-1}y(0) - \cdots - y^{(n-1)}(0)] \\ + a_{n-1}[s^{n-1}Y(s) - s^{n-2}y(0) - \cdots - y^{(n-2)}(0)] + \cdots + a_0 Y(s) = G(s)$$

where $\mathscr{L}\{y(t)\} = Y(s)$ and $\mathscr{L}\{g(t)\} = G(s)$.

In other words, *the Laplace transform of a linear differential equation with constant coefficients becomes an algebraic equation in $Y(s)$*.

Laplace Transform Diagram



Example 4: Solving a First-Order IVP

Use the Laplace transform to solve the initial-value problem

$$\frac{dy}{dt} + 3y = 13\sin 2t, \quad y(0) = 6$$

Example 5: Solving a Second-Order IVP

Solve

$$y'' - 3y' + 2y = e^{-4t}, \quad y(0) = 1,\ y'(0) = 5$$

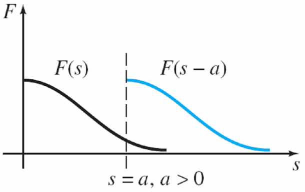
First Translation Theorem (1/2)

Theorem (7.3.1: First Translation Theorem)

If $\mathscr{L}\{f(t)\} = F(s)$ *and* a *is any real number, then*

$$\mathscr{L}\{e^{at}f(t)\} = F(s-a)$$

It is sometimes written as $\mathscr{L}\{e^{at}f(t)\} = \mathscr{L}\{f(t)\}|_{s\to s-a}$



(Multiplication in time domain implies translation in s domain.)

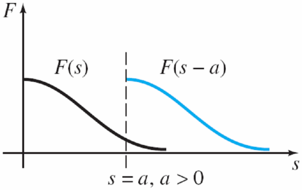
First Translation Theorem (2/2)

Remark (Inverse Transform)

The inverse transform of $F(s-a)$ *is given by*

$$\mathscr{L}^{-1}\{F(s-a)\} = \mathscr{L}^{-1}\{F(s)|_{s \to s-a}\} = e^{at}f(t)$$

where $f(t) = \mathscr{L}^{-1}\{F(s)\}$.



Example 1: Using the First Translation Theorem

Evaluate (a) $\mathscr{L}\left\{e^{5t}t^3\right\}$ (b) $\mathscr{L}\left\{e^{-2t}\cos 4t\right\}$.

Example 2: Partial Fraction: Repeated Linear Factors

Evaluate (a) $\mathscr{L}^{-1}\left\{\dfrac{2s+5}{(s-3)^2}\right\}$ (b) $\mathscr{L}^{-1}\left\{\dfrac{s/2+5/3}{s^2+4s+6}\right\}$.

Example 3: An Initial-Value Problem

Solve

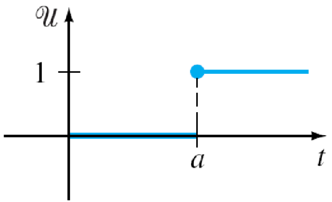
$$y'' - 6y' + 9y = t^2 e^{3t}, \quad y(0) = 2,\ y'(0) = 17$$

Unit Step Function

Definition (7.3.1: Unit Step Function)

The **unit step function** or **Heaviside function** $\mathscr{U}(t-a)$ is defined to be

$$\mathscr{U}(t-a) = \begin{cases} 0, & 0 \le t < a \\ 1, & t \ge a \end{cases}$$



Example 5: A Piecewise-Defined Function

Express

$$f(t) = \begin{cases} 20t, & 0 \le t < 5 \\ 0, & t \ge 5 \end{cases}$$

in terms of unit step functions. Graph.

Second Translation Theorem (1/2)

Theorem (7.3.2: Second Translation Theorem)

If $F(s) = \mathscr{L}\{f(t)\}$ *and* $a > 0$*, then*

$$\mathscr{L}\{f(t-a)\mathscr{U}(t-a)\} = e^{-as}F(s)$$

Proof.

$$\begin{aligned}
\mathscr{L}\{f(t-a)\mathscr{U}(t-a)\} &= \int_0^\infty e^{-st}f(t-a)\mathscr{U}(t-a)dt = \int_a^\infty e^{-st}f(t-a)dt \\
&= \int_0^\infty e^{-s(u+a)}f(u)du = e^{-as}\int_0^\infty e^{-su}f(u)du \\
&= e^{-as}\mathscr{L}\{f(t)\}
\end{aligned}$$

□

Second Translation Theorem (2/2)

Remark (Inverse Transform)

If $f(t) = \mathscr{L}^{-1}\{F(s)\}$ and $a > 0$, the inverse form is given by

$$\mathscr{L}^{-1}\{e^{-as}F(s)\} = f(t-a)\mathscr{U}(t-a) \tag{3}$$

Example 6: Using Formula (3)

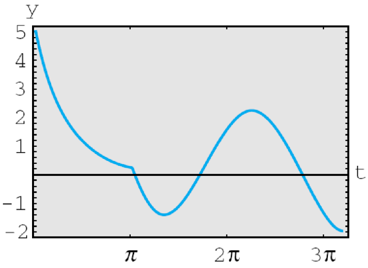
Evaluate (a) $\mathscr{L}^{-1}\left\{\frac{1}{s-4}e^{-2s}\right\}$ (b) $\mathscr{L}^{-1}\left\{\frac{s}{s^2+9}e^{-\frac{\pi s}{2}}\right\}$.

Example 7: Second Translation Theorem – Alterative Form

Evaluate $\mathscr{L}\{\cos t\, \mathscr{U}(t-\pi)\}$.

Example 8: An Initial-Value Problem

Solve $y' + y = f(t),\ y(0) = 5$, where $f(t) = \begin{cases} 0, & 0 \le t < \pi \\ 3\cos t, & t \ge \pi \end{cases}$



Derivatives of Transforms (1/2)

Theorem (7.4.1: Derivatives of Transforms)

If $F(s) = \mathscr{L}\{f(t)\}$ *and* $n = 1, 2, \ldots$, *then*

$$\mathscr{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} F(s)$$

For example,

$$\mathscr{L}\{tf(t)\} = -\frac{d}{ds}\mathscr{L}\{f(t)\}$$

Similarly,

$$\mathscr{L}\{t^2 f(t)\} = \frac{d^2}{ds^2}\mathscr{L}\{f(t)\}$$

Remark

By definition $F(s) = \int_0^\infty e^{-st} f(t)dt$

$$\mathscr{L}\{tf(t)\} = \int_0^\infty e^{-st} tf(t)dt$$

$$\frac{d}{ds}F(s) = \int_0^\infty (-t)e^{-st} f(t)dt = -\int_0^\infty e^{-st} tf(t)dt = -\mathscr{L}\{tf(t)\}$$ □

Example 1: Using Theorem 7.4.1

Evaluate $\mathscr{L}\{t \sin kt\}$.

Example 2: An Initial-Value Problem

Solve

$$x'' + 16x = \cos 4t, \; x(0) = 0, x'(0) = 1$$

Convolution (1/2)

If functions f and g are piecewise continuous on $[0, \infty)$, then a special product, denoted by $f * g$, is defined by the integral

$$f * g = \int_0^t f(\tau)g(t-\tau)d\tau$$

and is called the **convolution** of f and g.

The convolution of two functions is commutative: $f * g = g * f$. That is,

$$\int_0^t f(\tau)g(t-\tau)d\tau = \int_0^t g(\tau)f(t-\tau)d\tau$$

It is *not* true that the integral of a product of functions is the product of the integrals.

Convolution (2/2)

The convolution $f * g$ is a function of t.

Example

Verify that

$$e^t * \sin t = \int_0^t e^\tau \sin(t-\tau)d\tau = \frac{1}{2}(-\sin t - \cos t + e^t)$$

Convolution Theorem

Theorem (7.4.2: Convolution Theorem)

If $f(t)$ and $g(t)$ are piecewise continuous on $[0, \infty)$ and of exponential order, then

$$\mathscr{L}\{f * g\} = \mathscr{L}\{f(t)\}\mathscr{L}\{g(t)\} = F(s)G(s)$$

Likewise,

$$\mathscr{L}^{-1}\{F(s)G(s)\} = f * g$$

Proof.

Please check the textbook, page 308. □

Example 3: Transform of a Convolution

Evaluate $\mathscr{L}\left\{\int_0^t e^{\tau}\sin(t-\tau)d\tau\right\}$.

Example 4: Inverse Transform as a Convolution

Evaluate $\mathscr{L}^{-1}\left\{\dfrac{1}{(s^2+k^2)^2}\right\}$.

Transform of an Integral

When $g(t) = 1$ and $\mathscr{L}\{g(t)\} = G(s) = 1/s$, the convolution theorem implies that the Laplace transform of the integral f is

$$\mathscr{L}\left\{\int_0^t f(\tau)d\tau\right\} = \mathscr{L}\left\{\int_0^t 1 \cdot f(\tau)d\tau\right\} = \mathscr{L}\left\{\int_0^t g(t-\tau)\cdot f(\tau)d\tau\right\}$$
$$= \mathscr{L}\{g(t-\tau) = 1\} \cdot \mathscr{L}\{f(t)\} = \frac{F(s)}{s}$$

The corresponding inverse form

$$\int_0^t f(\tau)d\tau = \mathscr{L}^{-1}\left\{\frac{F(s)}{s}\right\}$$

can be used in lieu of partial fractions when s^n is a factor of the denominator and $f(t) = \mathscr{L}^{-1}\{F(s)\}$ is easy to integrate.

Example

Given $f(t) = \sin t$ and $F(s) = 1/(s^2+1)$, find

(a) $\mathscr{L}^{-1}\left\{\dfrac{1}{s(s^2+1)}\right\}$

(b) $\mathscr{L}^{-1}\left\{\dfrac{1}{s^2(s^2+1)}\right\}$

(c) $\mathscr{L}^{-1}\left\{\dfrac{1}{s^3(s^2+1)}\right\}$.

Example 5: An Integral Equation

Solve

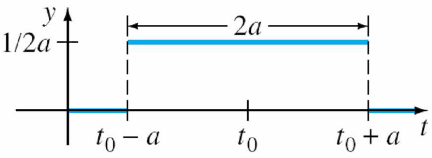
$$f(t) = 3t^2 - e^{-t} - \int_0^t f(\tau) e^{t-\tau} d\tau \quad \text{for } f(t)$$

Unit Impulse Function

The function

$$\delta_a(t-t_0) = \begin{cases} 0, & 0 \le t < t_0 - a \\ \frac{1}{2a}, & t_0 - a \le t < t_0 + a \\ 0, & t \ge t_0 + a \end{cases}$$

is called a **unit impulse**, because it possesses the integration property $\displaystyle\int_0^\infty \delta_a(t-t_0)dt = 1$.



(a) graph of $\delta_a(t - t_0)$

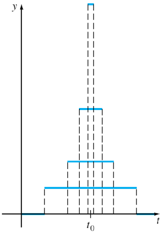
Dirac Delta Function

A "function" that approximates $\delta_a(t-t_0)$ and is defined by the limit

$$\delta(t-t_0) = \lim_{a\to 0} \delta_a(t-t_0)$$

is called the **Dirac delta function.** It has the following two properties:

$$\delta(t-t_0) = \begin{cases} \infty, & t = t_0 \\ 0, & t \neq t_0 \end{cases} \qquad \text{and} \qquad \int_0^\infty \delta(t-t_0)dt = 1$$



(b) behavior of δ_a as $a \to 0$

Transform of Dirac Delta Function

Theorem (7.5.1: Transform of Dirac Delta Function)

For $t_0 > 0$,

$$\mathscr{L}\left\{\delta(t-t_0)\right\} = e^{-st_0}$$

Furthermore,

$$\mathscr{L}\left\{\delta(t)\right\} = 1$$

Proof.

By definition. □

Example 1: Two Initial-Value Problem

Solve $y'' + y = 4\delta(t - 2\pi)$ subject to

(a) $y(0) = 1, y'(0) = 0$

(b) $y(0) = 0, y'(0) = 0$

Systems of Linear Differential Equations

Solve

$$\begin{aligned} x_1'' + 10x_1 - 4x_2 &= 0 \\ -4x_1 + x_2'' + 4x_2 &= 0 \end{aligned}$$

subject to $x_1(0) = 0, x_1'(0) = 1, x_2(0) = 0, x_2'(0) = -1$.

Homework

- Exercises 7.1: 4, 13, 26, 37.
- Exercises 7.2: 9, 24, 33, 38, 41.
- Exercises 7.3: 5, 16, 23, 32, 39, 55, 66.
- Exercises 7.4: 5, 10, 27, 41, 50.
- Exercises 7.5: 5, 12.
- Exercises 7.6: 4.