Differential Equations

Lecture Set 08
Systems of Linear First-Order Differential Equations

林惠勇
Huei-Yung Lin

lin@ee.ccu.edu.tw

Robot Vision Lab
Department of Electrical Engineering
National Chung Cheng University
Chiayi 621, Taiwan

First-Order System

A system of n first-order differential equations

$$\begin{aligned}
\frac{dx_1}{dt} &= g_1(t, x_1, x_2, \ldots, x_n) \\
\frac{dx_2}{dt} &= g_2(t, x_1, x_2, \ldots, x_n) \\
&\vdots \\
\frac{dx_n}{dt} &= g_n(t, x_1, x_2, \ldots, x_n)
\end{aligned}$$

is called a **first-order system.**

Linear Systems (1/2)

When each of the functions $g_1, g_2, \ldots, g_n$ in the first-order system is *linear* in the dependent variables $x_1, x_2, \ldots, x_n$, we get the **normal form** of a first-order system of linear equations:[1]

$$\begin{aligned}
\frac{dx_1}{dt} &= a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n + f_1(t) \\
\frac{dx_2}{dt} &= a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n + f_2(t) \\
&\vdots \\
\frac{dx_n}{dt} &= a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n + f_n(t)
\end{aligned}$$

This system is referred simply as a **linear system.**

[1] a_{ij} and x_i are functions of t.

Linear Systems (2/2)

We assume the coefficients a_{ij} as well as the functions f_i are continuous on a common interval I.

When $f_i(t) = 0, i = 1, 2, \ldots, n$, the linear system is said to be **homogeneous**; otherwise it is **nonhomogeneous**.

Matrix Form of a Linear system

If $\mathbf{X}$, $\mathtt{A}(t)$, and $\mathbf{F}(t)$ denote the respective matrices

$$\mathbf{X} = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix}, \; \mathtt{A}(t) = \begin{pmatrix} a_{11}(t) & a_{12}(t) & \cdots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \cdots & a_{2n}(t) \\ \vdots & & \vdots & \\ a_{n1}(t) & a_{n2}(t) & \cdots & a_{nn}(t) \end{pmatrix}, \; \mathbf{F}(t) = \begin{pmatrix} f_1(t) \\ f_2(t) \\ \vdots \\ f_n(t) \end{pmatrix}$$

then the system of linear 1st-order DE can be written as

$$\mathbf{X}' = \mathtt{A}\mathbf{X} + \mathbf{F}$$

If the system is homogeneous, its matrix form is then

$$\mathbf{X}' = \mathtt{A}\mathbf{X}$$

Example 1: Systems Written in Matrix Notation

Rewrite the system of linear 1-st order DEs in matrix notation.

$$\frac{dx}{dt} = 6x + y + z + t$$
$$\frac{dy}{dt} = 8x + 7y - z + 10t$$
$$\frac{dz}{dt} = 2x + 9y - z + 6t$$

Solution Vector (1/2)

Definition (8.1.1: Solution Vector)

A **solution vector** on an interval I is any column vector

$$\mathbf{X} = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix}$$

whose entries are differentiable functions satisfying the system

$$\mathbf{X}' = \mathbb{A}\mathbf{X} + \mathbf{F}$$

on the interval.

Solution Vector (2/2)

Remark

In the important case $n = 2$, the equations $x_1(t)$ and $x_2(t)$ represent a curve in the x_1x_2-plane. It is common practice to call a curve in the plane a **trajectory** *and call the x_1x_2-plane the* **phase plane**.

Example 2: Verification of Solutions

Verify that on the interval $(-\infty, \infty)$

$$\mathbf{X}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-2t} \quad \text{and} \quad \mathbf{X}_2 = \begin{pmatrix} 3 \\ 5 \end{pmatrix} e^{6t}$$

are solutions of

$$\mathbf{X}' = \begin{pmatrix} 1 & 3 \\ 5 & 3 \end{pmatrix} \mathbf{X}$$

Initial-Value Problem

Let t_0 denote a point on an interval I and

$$\mathbf{X}(t_0) = \begin{pmatrix} x_1(t_0) \\ x_2(t_0) \\ \vdots \\ x_n(t_0) \end{pmatrix} \quad \text{and} \quad \mathbf{X}_0 = \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \vdots \\ \gamma_n \end{pmatrix}$$

where $\gamma_i, i = 1, 2, \ldots, n$ are given constants. Then the problem

Solve:

$$\mathbf{X}' = \mathbb{A}(t)\mathbf{X} + \mathbf{F}(t)$$

Subject to:

$$\mathbf{X}(t_0) = \mathbf{X}_0$$

is an **initial-value problem** on the interval.

Existence of a Unique Solution

Theorem (8.1.1: Existence of a Unique Solution)

Let the entries of the matrices $\mathbb{A}(t)$ and $\mathbf{F}(t)$ be functions continuous on a common interval I that contains the point t_0. Then there exists a unique solution of the initial-value problem on the interval.

Superposition Principle

Theorem (8.1.2: Superposition Principle)

Let $\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_k$ *be a set of solution vectors of the homogeneous system*

$$\mathbf{X}' = \mathtt{A}\mathbf{X}$$

on an interval I*. Then the linear combination*

$$\mathbf{X} = c_1\mathbf{X}_1 + c_2\mathbf{X}_2 + \cdots + c_k\mathbf{X}_k$$

where $c_i, i = 1, 2, \ldots, k$ *are arbitrary constants, is also a solution on the interval.*

Example 3: Using the Superposition Principle

Verify that

$$\mathbf{X}_1 = \begin{pmatrix} \cos t \\ -\frac{1}{2}\cos t + \frac{1}{2}\sin t \\ -\cos t - \sin t \end{pmatrix}, \quad \mathbf{X}_2 = \begin{pmatrix} 0 \\ e^t \\ 0 \end{pmatrix}, \quad \mathbf{X}_3 = c_1\mathbf{X}_1 + c_2\mathbf{X}_2$$

are solutions of the system

$$\mathbf{X}' = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ -2 & 0 & -1 \end{pmatrix}\mathbf{X}$$

Linear Dependence/Independence

Definition (8.1.2: Linear Dependence/Independence)

Let $\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_k$ be a set of solution vectors of the homogeneous system

$$\mathbf{X}' = \mathtt{A}\mathbf{X}$$

on an interval I. We say that the set is **linearly dependent** on the interval if there exist constants $c_1, c_2, \ldots, c_k$, not all zero, such that

$$c_1\mathbf{X}_1 + c_2\mathbf{X}_2 + \cdots + c_k\mathbf{X}_k = \mathbf{0}$$

for every t in the interval. If the set of vectors is not linearly dependent on the interval, it is said to be **linearly independent**.

Criterion for Linear Independent Solutions

Theorem (8.1.3: Criterion for Linear Independent Solutions)

Let $\mathbf{X}_1 = \begin{pmatrix} x_{11} \\ x_{21} \\ \vdots \\ x_{n1} \end{pmatrix}, \quad \mathbf{X}_2 = \begin{pmatrix} x_{12} \\ x_{22} \\ \vdots \\ x_{n2} \end{pmatrix}, \quad \ldots, \quad \mathbf{X}_n = \begin{pmatrix} x_{1n} \\ x_{2n} \\ \vdots \\ x_{nn} \end{pmatrix}$ *be* n *solution vectors of the homogeneous system* $\mathbf{X}' = \mathbb{A}\mathbf{X}$ *on an interval* I*. Then the set of solution vectors is linearly independent on* I *if and only if the* **Wronskian**

$$W(\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_n) = \begin{vmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & & & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{vmatrix} \neq 0$$

for every t *in the interval.*

Wronskian

It can be shown that if $\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_n$ are solution vectors of

$$\mathbf{X}' = \mathtt{A}\mathbf{X}$$

then for every t in I either $W(\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_n) \neq 0$ or $W(\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_n) = 0$.

Thus if we can show that $W \neq 0$ for some t_0 in I, then $W \neq 0$ for every t, and hence the solutions are linearly independent on the interval.

Example 4: Linearly Independent Solutions

Verify that the solutions

$$\mathbf{X}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-2t} \quad \text{and} \quad \mathbf{X}_2 = \begin{pmatrix} 3 \\ 5 \end{pmatrix} e^{6t}$$

of the system

$$\mathbf{X}' = \begin{pmatrix} 1 & 3 \\ 5 & 3 \end{pmatrix} \mathbf{X}$$

given in Slide (9) are linearly independent solutions.

Existence of a Fundamental Set of Solutions

Theorem (8.1.4: Existence of a Fundamental Set)

Any set $\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_n$ of n linearly independent solution vectors of the homogeneous system

$$\mathbf{X}' = \mathtt{A}\mathbf{X}$$

on an interval I is said to be a **fundamental set of solutions** *on the interval.*

There exists a fundamental set of solutions for the homogeneous system

$$\mathbf{X}' = \mathtt{A}\mathbf{X}$$

on an interval I.

General Solution – Homogeneous Systems

Theorem (8.1.5: General Solution – Homogeneous Systems)

Let $\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_n$ be a fundamental set of solutions of the homogeneous system

$$\mathbf{X}' = \mathtt{A}\mathbf{X}$$

on an interval I. Then the **general solution** *of the system on the interval is*

$$\mathbf{X} = c_1\mathbf{X}_1 + c_2\mathbf{X}_2 + \cdots + c_n\mathbf{X}_n$$

where $c_i, i = 1, 2, \ldots, n$ are arbitrary constants.

Example 5: General Solution

Find the general solution of

$$\mathbf{X}' = \begin{pmatrix} 1 & 3 \\ 5 & 3 \end{pmatrix} \mathbf{X}$$

Example 6: General Solution

Find the general solution of

$$\mathbf{X}' = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ -2 & 0 & -1 \end{pmatrix} \mathbf{X}$$

General Solution – Nonhomogeneous Systems

Theorem (8.1.6: General Solution – Nonhomogeneous Systems)

Let $\mathbf{X}_p$ be a given solution of the nonhomogeneous system

$$\mathbf{X}' = \mathtt{A}\mathbf{X} + \mathbf{F} \tag{1}$$

on an interval I, and let $\mathbf{X}_c = c_1\mathbf{X}_1 + c_2\mathbf{X}_2 + \cdots + c_n\mathbf{X}_n$ denote the general solution on the same interval of the associated homogeneous system

$$\mathbf{X}' = \mathtt{A}\mathbf{X} \tag{2}$$

Then the **general solution** *of the nonhomogeneous system on the interval is $\mathbf{X} = \mathbf{X}_c + \mathbf{X}_p$.*

The general solution $\mathbf{X}_c$ of the associated homogeneous system (2) is called the **complementary function** *of the nonhomogeneous system (1).*

Example 7: General Solution – Nonhomogeneous System

Find the general solution of

$$\mathbf{X}' = \begin{pmatrix} 1 & 3 \\ 5 & 3 \end{pmatrix} \mathbf{X} + \begin{pmatrix} 12t - 11 \\ -3 \end{pmatrix}$$

Eigenvalues and Eigenvectors (1/2)

Let

$$\mathbf{X} = (k_1 \;\; k_2 \;\; \cdots \;\; k_n)^\top e^{\lambda t} = \mathbf{K}e^{\lambda t}$$

be a *solution vector* of the general homogeneous linear first-order system

$$\mathbf{X}' = \mathtt{A}\mathbf{X} \tag{3}$$

then

$$\lambda \mathbf{K} e^{\lambda t} = \mathtt{A}\mathbf{K}e^{\lambda t}$$

That is,

$$(\mathtt{A} - \lambda \mathtt{I})\mathbf{K} = 0 \tag{4}$$

Note: The symbol $^\top$ is a transpose, and $(k_1 \;\; k_2 \;\; \cdots \;\; k_n)^\top$ is a column vector in a more compact form.

Eigenvalues and Eigenvectors (2/2)

If the system has a nontrivial solution, then we must have

$$\det(\mathtt{A} - \lambda \mathtt{I}) = 0$$

The polynomial equation in λ is called the **characteristic equation** of the matrix $\mathtt{A}$; its solutions are the **eigenvalues** of $\mathtt{A}$.

A solution $\mathbf{K} \neq \mathbf{0}$ of Eq. (4) corresponding to an eigenvalue λ is called an **eigenvector** of $\mathtt{A}$.

A solution of the homogeneous system[2]

$$\mathbf{X}' = \mathtt{A}\mathbf{X}$$

is then $\mathbf{X} = \mathbf{K}e^{\lambda t}$.

[2] Plug in the solution to the homogeneous system to verify!

General Solution – Homogeneous Systems

Theorem (8.2.1: General Solution – Homogeneous Systems)

Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ *be* n *distinct real eigenvalues of the coefficient matrix* $\mathtt{A}$ *of the homogeneous system*

$$\mathbf{X}' = \mathtt{A}\mathbf{X}$$

and let $\mathbf{K}_1, \mathbf{K}_2, \ldots, \mathbf{K}_n$ *be the corresponding eigenvectors.*

Then the **general solution** *of*

$$\mathbf{X}' = \mathtt{A}\mathbf{X}$$

on the interval $(-\infty, \infty)$ *is given by*

$$\mathbf{X} = c_1\mathbf{K}_1 e^{\lambda_1 t} + c_2\mathbf{K}_2 e^{\lambda_2 t} + \cdots + c_n\mathbf{K}_n e^{\lambda_n t}$$

Example 1: Distinct Eigenvalues

Solve

$$\frac{dx}{dt} = 2x + 3y$$
$$\frac{dy}{dt} = 2x + y$$

Example 2: Distinct Eigenvalues

Solve

$$\frac{dx}{dt} = -4x + y + z$$
$$\frac{dy}{dt} = x + 5y - z$$
$$\frac{dz}{dt} = y - 3z$$

Repeated Eigenvalues (1/2)

Not all of the n eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ for an $n \times n$ matrix $\mathbb{A}$ need be distinct.

In general, if m is a positive integer and $(\lambda - \lambda_1)^m$ is a factor of the characteristic equation while $(\lambda - \lambda_1)^{m+1}$ is not a factor, then λ_1 is said to be an **eigenvalue of multiplicity** m.

- For some $n \times n$ matrices $\mathbb{A}$ it may be possible to find m linearly independent eigenvectors $\mathbf{K}_1, \mathbf{K}_2, \ldots, \mathbf{K}_m$ corresponding to an eigenvalue λ_1 of multiplicity $m \leq n$. In this case the general solution of the system contains the linear combination

$$c_1\mathbf{K}_1 e^{\lambda_1 t} + c_2\mathbf{K}_2 e^{\lambda_1 t} + \cdots + c_m\mathbf{K}_m e^{\lambda_1 t}$$

Repeated Eigenvalues (2/2)

- If there is only one eigenvector corresponding to the eigenvalue λ_1 of multiplicity m, then m linearly independent solution of the form

$$
\begin{aligned}
\mathbf{X}_1 &= \mathbf{K}_{11} e^{\lambda_1 t} \\
\mathbf{X}_2 &= \mathbf{K}_{21} t e^{\lambda_1 t} + \mathbf{K}_{22} e^{\lambda_1 t} \\
&\cdots \quad \cdots \\
&\cdots \quad \cdots \\
\mathbf{X}_m &= \mathbf{K}_{m1} \frac{t^{m-1}}{(m-1)!} e^{\lambda_1 t} + \mathbf{K}_{m2} \frac{t^{m-2}}{(m-2)!} e^{\lambda_1 t} + \cdots + \mathbf{K}_{mm} e^{\lambda_1 t}
\end{aligned}
$$

where $\mathbf{K}_{ij}$ are *column vectors*, can always be found.

Example 3: Repeated Eigenvalues

Solve

$$\mathbf{X}' = \begin{pmatrix} 1 & -2 & 2 \\ -2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix} \mathbf{X}$$

Second Solution

Suppose that λ_1 is an eigenvalue of multiplicity two and that there is only one eigenvector associated with this value. A second solution can be found of the form

$$\mathbf{X}_2 = \mathbf{K}te^{\lambda_1 t} + \mathbf{P}e^{\lambda_1 t} \tag{5}$$

where

$$\mathbf{K} = \begin{pmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{pmatrix} \quad \text{and} \quad \mathbf{P} = \begin{pmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \end{pmatrix}$$

Example 4: Repeated Eigenvalues

Find the general solution of the system

$$\mathbf{X}' = \begin{pmatrix} 3 & -18 \\ 2 & -9 \end{pmatrix} \mathbf{X}$$

Note: Do not use Eqs. (13) and (14) in the textbook!!!

Eigenvalue of Multiplicity Three

When the coefficient matrix $\mathbb{A}$ has only one eigenvector associated with an eigenvalue λ_1 of multiplicity three, we can find a second solution of the form (5) and a third solution of the form

$$\mathbf{X}_3 = \mathbf{K}\frac{t^2}{2}e^{\lambda_1 t} + \mathbf{P}te^{\lambda_1 t} + \mathbf{Q}e^{\lambda_1 t} \tag{6}$$

where

$$\mathbf{K} = \begin{pmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{pmatrix}, \qquad \mathbf{P} = \begin{pmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \end{pmatrix}, \qquad \mathbf{Q} = \begin{pmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \end{pmatrix}$$

Example

Verify that the general solution of

$$\frac{dx}{dt} = 6x - y$$
$$\frac{dy}{dt} = 5x + 4y$$

is

$$\mathbf{X} = c_1 \begin{pmatrix} 1 \\ 1 - 2i \end{pmatrix} e^{(5+2i)t} + c_2 \begin{pmatrix} 1 \\ 1 + 2i \end{pmatrix} e^{(5-2i)t}$$

Solutions Corresponding to a Complex Eigenvalue

Theorem (8.2.2: Solutions Corresponding to a Complex Eigenvalue)

Let $\mathtt{A}$ *be the coefficient matrix having real entries of the homogeneous system*

$$\mathbf{X}' = \mathtt{A}\mathbf{X}$$

and let $\mathbf{K}_1$ *be an eigenvector corresponding to the complex eigenvalue* $\lambda_1 = \alpha + i\beta$, *where* α *and* β *are real.*

Then

$$\mathbf{K}_1 e^{\lambda_1 t} \quad \text{and} \quad \overline{\mathbf{K}}_1 e^{\overline{\lambda}_1 t}$$

are solutions of the system $\mathbf{X}' = \mathtt{A}\mathbf{X}$.

Remark

Check the previous example!

Real Solutions Corresp. to a Complex Eigenvalue

Theorem (8.2.3: Real Sol. Corresp. to a Complex Eigenvalue)

Let $\lambda_1 = \alpha + i\beta$ *be a complex eigenvalue of the coefficient matrix* $\mathtt{A}$ *in the homogeneous system* $\mathbf{X}' = \mathtt{A}\mathbf{X}$*, and let* $\mathbf{B}_1$ *and* $\mathbf{B}_2$ *denote the column vectors defined by*

$$\mathbf{B}_1 = Re(\mathbf{K}_1) \quad \text{and} \quad \mathbf{B}_2 = Im(\mathbf{K}_1)$$

Then

$$\mathbf{X}_1 = e^{\alpha t}[\mathbf{B}_1 \cos\beta t - \mathbf{B}_2 \sin\beta t]$$
$$\mathbf{X}_2 = e^{\alpha t}[\mathbf{B}_2 \cos\beta t + \mathbf{B}_1 \sin\beta t]$$

are linearly independent solution of $\mathbf{X}' = \mathtt{A}\mathbf{X}$ *on* $(-\infty, \infty)$.

Note: Recall the Euler's formula: $e^{i\theta} = \cos\theta + i\sin\theta$.
(Check page 350 in the textbook for proof.)

Example 6: Complex Eigenvalues

Solve the initial-value problem

$$\mathbf{X}' = \begin{pmatrix} 2 & 8 \\ -1 & -2 \end{pmatrix} \mathbf{X}, \quad \mathbf{X}(0) = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

Nonhomogeneous Linear Systems

- The methods of **undetermined coefficients** and **variation of parameters** used to find particular solutions of nonhomogeneous linear ODEs can both be adapted to the solution of nonhomogeneous linear systems.
- Of the two methods, *variation of parameters* is the more powerful technique. However, there are instances when the method of *undetermined coefficients* provides a quick means of finding a particular solution.

Undetermined Coefficients

The **method of undetermined coefficients** consists of making an *educated guess* about the form of a particular solution vector $\mathbf{X}_p$; the guess is motivated by the types of functions that make up the entries of the column matrix $\mathbf{F}(t)$.

The matrix version of undetermined coefficients is applicable to

$$\mathbf{X}' = \mathtt{A}\mathbf{X} + \mathbf{F}(t)$$

only when the entries of $\mathtt{A}$ are constants and the entries of $\mathbf{F}(t)$ are constants, polynomials, exponential functions, sines and cosines, or finite sums and products of these functions.

Example 1: Undetermined Coefficients

Solve the system

$$\mathbf{X}' = \begin{pmatrix} -1 & 2 \\ -1 & 1 \end{pmatrix} \mathbf{X} + \begin{pmatrix} -8 \\ 3 \end{pmatrix}$$

on $(-\infty, \infty)$

Example 3: Form of $\mathbf{X}_p$

Determine the form of a particular solution vector $\mathbf{X}_p$ for the system

$$\begin{aligned} \frac{dx}{dt} &= 5x + 3y - 2e^{-t} + 1 \\ \frac{dy}{dt} &= -x + y + e^{-t} - 5t + 7 \end{aligned}$$

A Fundamental Matrix (1/2)

If $\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_n$ is a fundamental set of solutions of the homogeneous system

$$\mathbf{X}' = \mathtt{A}\mathbf{X}$$

on an interval I, then its general solution on the interval is the linear combination $\mathbf{X} = c_1\mathbf{X}_1 + c_2\mathbf{X}_2 + \cdots + c_n\mathbf{X}_n$ or

$$\mathbf{X} = c_1 \begin{pmatrix} x_{11} \\ x_{21} \\ \vdots \\ x_{n1} \end{pmatrix} + c_2 \begin{pmatrix} x_{12} \\ x_{22} \\ \vdots \\ x_{n2} \end{pmatrix} + \cdots + c_n \begin{pmatrix} x_{1n} \\ x_{2n} \\ \vdots \\ x_{nn} \end{pmatrix}$$

$$= \begin{pmatrix} c_1x_{11} + c_2x_{12} + \cdots + c_nx_{1n} \\ c_1x_{21} + c_2x_{22} + \cdots + c_nx_{2n} \\ \vdots \\ c_1x_{n1} + c_2x_{n2} + \cdots + c_nx_{nn} \end{pmatrix}$$

A Fundamental Matrix (2/2)

It can also be written as

$$\mathbf{X} = \Phi(t)\mathbf{C}$$

where $\mathbf{C}$ is an $n \times 1$ column vector of arbitrary constants, $c_1, c_2, \cdots, c_n$, and the $n \times n$ matrix

$$\Phi(t) = \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & & & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{pmatrix}$$

is called a **fundamental matrix** of the system in the interval.

Properties of Fundamental Matrix

There are some important properties of the fundamental matrix:

- A fundamental matrix $\Phi(t)$ is *nonsingular*.
- The inverse $\Phi^{-1}(t)$ exists for every t in the interval.
- If $\Phi(t)$ is a fundamental matrix of the system $\mathbf{X}' = \mathtt{A}\mathbf{X}$, then

$$\Phi'(t) = \mathtt{A}\Phi(t) \tag{7}$$

Variation of Parameters (1/2)

Check if it is possible to replace the matrix of constants $\mathbf{C}$ in $\mathbf{X} = \Phi(t)\mathbf{C}$ by a column vector of functions $\mathbf{U}(t) = \begin{pmatrix} u_1(t) & u_2(t) & \cdots & u_n(t) \end{pmatrix}^\top$ so that

$$\mathbf{X}_p = \Phi(t)\mathbf{U}(t) \tag{8}$$

is a particular solution of the nonhomogeneous system[3]

$$\mathbf{X}' = \mathbb{A}\mathbf{X} + \mathbf{F}(t) \tag{9}$$

From Eq. (8), $\Rightarrow \mathbf{X}_p' = \Phi(t)\mathbf{U}'(t) + \Phi'(t)\mathbf{U}(t)$
From Eq. (9), $\Rightarrow \Phi(t)\mathbf{U}'(t) + \Phi'(t)\mathbf{U}(t) = \mathbb{A}\Phi(t)\mathbf{U}(t) + \mathbf{F}(t)$
From Eq. (7), $\Rightarrow \Phi(t)\mathbf{U}'(t) + \mathbb{A}\Phi(t)\mathbf{U}(t) = \mathbb{A}\Phi(t)\mathbf{U}(t) + \mathbf{F}(t)$
$\Rightarrow \Phi(t)\mathbf{U}'(t) = \mathbf{F}(t) \Rightarrow \mathbf{U}'(t) = \Phi^{-1}(t)\mathbf{F}(t) \Rightarrow \mathbf{U}(t) = \int \Phi^{-1}(t)\mathbf{F}(t)dt$
Thus, $\mathbf{X}_p = \Phi(t)\mathbf{U}(t) = \Phi(t)\int \Phi^{-1}(t)\mathbf{F}(t)dt$ □

[3] Check Chapter 4, Slide 61.

Variation of Parameters (2/2)

A particular solution of $\mathbf{X}' = \mathbb{A}\mathbf{X} + \mathbf{F}(t)$ is

$$\mathbf{X}_p = \Phi(t) \int \Phi^{-1}(t)\mathbf{F}(t)dt$$

The general solution is $\mathbf{X} = \mathbf{X}_c + \mathbf{X}_p$, or

$$\mathbf{X} = \Phi(t)\mathbf{C} + \Phi(t) \int \Phi^{-1}(t)\mathbf{F}(t)dt$$

Example 4: Variation of Parameters

Solve the system

$$\mathbf{X}' = \begin{pmatrix} -3 & 1 \\ 2 & -4 \end{pmatrix} \mathbf{X} + \begin{pmatrix} 3t \\ e^{-t} \end{pmatrix}$$

on $(-\infty, \infty)$

Matrix Exponential (1/2)

Definition (8.4.1: Matrix Exponential)

For any $n \times n$ matrix $\mathtt{A}$, the **matrix exponential** is defined as

$$e^{\mathtt{A}t} = \mathtt{I} + \mathtt{A}t + \mathtt{A}^2 \frac{t^2}{2!} + \cdots + \mathtt{A}^k \frac{t^k}{k!} + \cdots = \sum_{k=0}^{\infty} \mathtt{A}^k \frac{t^k}{k!}$$

Remark

The **derivative of the matrix exponential** *is given by*[a]

$$\frac{d}{dt}e^{\mathtt{A}t} = \mathtt{A}e^{\mathtt{A}t}$$

Thus,

$$\mathbf{X} = e^{\mathtt{A}t}\mathbf{C}$$

is a solution of $\mathbf{X}' = \mathtt{A}\mathbf{X}$.

The function $\Psi(t) = e^{\mathtt{A}t}$ *is a fundamental matrix of the system* $\mathbf{X}' = \mathtt{A}\mathbf{X}$.

[a] By definition 8.4.1.

Nonhomogeneous Systems

For a nonhomogeneous system of linear first-order differential equations

$$\mathbf{X}' = \mathbb{A}\mathbf{X} + \mathbf{F}(t)$$

where $\mathbb{A}$ is an $n \times n$ matrix of constants, the general solution is given by

$$\mathbf{X} = \mathbf{X}_c + \mathbf{X}_p = e^{\mathbb{A}t}\mathbf{C} + e^{\mathbb{A}t}\int_{t_0}^{t} e^{-\mathbb{A}\tau}\mathbf{F}(\tau)d\tau$$

Computation of $e^{\mathtt{A}t}$

The matrix $e^{\mathtt{A}t}$ can be computed by the Laplace transform:

$$e^{\mathtt{A}t} = \mathscr{L}^{-1}\left\{(s\mathtt{I} - \mathtt{A})^{-1}\right\}$$

since it is a solution of the initial-value problem

$$\mathbf{X}' = \mathtt{A}\mathbf{X} \quad \text{and} \quad \mathbf{X}(0) = \mathtt{I} \tag{10}$$

$\mathbf{x}(s) = \mathscr{L}\{\mathbf{X}(t)\} = \mathscr{L}\left\{e^{\mathtt{A}t}\right\}$

From the IVP Eq. (10) $\Rightarrow s\mathbf{x}(s) - \mathbf{X}(0) = \mathtt{A}\mathbf{x}(s)$

$\Rightarrow (s\mathtt{I} - \mathtt{A})\mathbf{x}(s) = \mathtt{I}$

$\Rightarrow \mathbf{x}(s) = (s\mathtt{I} - \mathtt{A})^{-1}\mathtt{I} = (s\mathtt{I} - \mathtt{A})^{-1}$

$\Rightarrow \mathscr{L}\left\{e^{\mathtt{A}t}\right\} = (s\mathtt{I} - \mathtt{A})^{-1}$ or $e^{\mathtt{A}t} = \mathscr{L}^{-1}\left\{(s\mathtt{I} - \mathtt{A})^{-1}\right\}$

Example 1: Matrix Exponential

Use the Laplace transform to compute $e^{\mathbb{A}t}$ for

$$\mathbb{A} = \begin{pmatrix} 1 & -1 \\ 2 & -2 \end{pmatrix}$$

Homework

- Exercises 8.1: 4, 13, 22.
- Exercises 8.2: 7, 14, 25, 31, 44.
- Exercises 8.3: 5, 16, 21, 32.
- Exercises 8.4: 4, 11, 22.

Appendix: Review of Eigensystem (1/2)

Definition

The *eigenvalues* of a $p \times p$ real or complex matrix $\mathtt{A}$ are the real or complex numbers λ for which there is a *nonzero* $\mathbf{x} \neq \mathbf{0}$ with $\mathtt{A}\mathbf{x} = \lambda\mathbf{x}$.

The *eigenvectors* of $\mathtt{A}$ are the *nonzero* vectors $\mathbf{x} \neq \mathbf{0}$ for which there is a number λ with $\mathtt{A}\mathbf{x} = \lambda\mathbf{x}$. If $\mathtt{A}\mathbf{x} = \lambda\mathbf{x}$ for $\mathbf{x} \neq \mathbf{0}$, then $\mathbf{x}$ is an eigenvector *associated with* the eigenvalue λ, and *vice versa*.

The associated eigenvalues and eigenvectors together make up the *eigensystem* of $\mathtt{A}$.

Appendix: Review of Eigensystem (2/2)

Theorem

λ is an eigenvalue of $\mathtt{A}$ if and only if $\mathtt{A} - \lambda\mathtt{I}$ is singular, which in turn holds if and only if the determinant of $\mathtt{A} - \lambda\mathtt{I}$ equals zero: $\det(\mathtt{A} - \lambda\mathtt{I}) = 0$ (the so-called characteristic equation of $\mathtt{A}$).