# Differential Equations

## Lecture Set 09
## Numerical Solutions of Ordinary Differential Equations

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# Truncation Errors for Euler's Method (1/3)

The **Euler's method** for numerical solution of the 1st-order IVP $y' = f(x, y)$, $y(x_0) = y_0$ is given by

$$y_{n+1} = y_n + hf(x_n, y_n) \tag{1}$$

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The **local truncation error** for $y_{n+1}$ defined by

$$y(x_{n+1}) - y_{n+1}$$

can be derived from the Taylor's expansion as

$$y(x_{n+1}) = y_n + hf(x_n, y_n) + y''(c)\frac{h^2}{2!} = y_{n+1} + y''(c)\frac{h^2}{2!}$$

where $c \in (x_n, x_{n+1})$.

# Truncation Errors for Euler's Method (2/3)

Local truncation error is also called *formula error*, or *discretization error*.

Hence, the local truncation error in $y_{n+1}$ is

$$y''(c)\frac{h^2}{2!}, \quad \text{where} \quad x_n < c < x_{n+1}$$

An upper bound on the absolute value of the error is $Mh^2/2!$ , where

$$M = \max_{x_n < x < x_{n+1}} |y''(x)|$$

# Truncation Errors for Euler's Method (3/3)

## Remark

*Taylor's formula with remainder:*

$$y(x) = y(a) + y'(a)\frac{x-a}{1!} + \cdots + y^{(k)}(a)\frac{(x-a)^k}{k!} + y^{(k+1)}(c)\frac{(x-a)^{k+1}}{(k+1)!}$$

*where* $c \in (a, x)$.

# Big O Notation

In discussing errors arising from numerical methods, it is helpful to use the notation $O(h^n)$ .

Let $e(h)$ denote the error in a numerical calculation depending on $h$. Then $e(h)$ is said to be of order $h^n$, denoted by $O(h^n)$, if there exist a constant $C$ and a positive integer $n$ such that $|e(h)| \leq Ch^n$ for $h$ sufficiently small.

Thus, the local truncation error for Euler's method is $O(h^2)$.

**Remark**

*In general, if $e(h)$ in a numerical method is of order $h^n$ and $h$ is halved, the new error is approximately $C(h/2)^n = Ch^n/2^n$; that is, the error is reduced by a factor of $1/2^n$.*

# Example 1: Bound for Local Truncation Errors

Find a bound for the local truncation errors for Euler's method applied to

$$y' = 2xy, \; y(1) = 1$$

# Improved Euler's Method (1/2)

The numerical method defined by the formula

$$y_{n+1} = y_n + h\frac{f(x_n, y_n) + f(x_{n+1}, y^*_{n+1})}{2} \tag{2}$$

where

$$y^*_{n+1} = y_n + hf(x_n, y_n) \tag{3}$$

is commonly known as the **improved Euler's method.**

**Remark**

*Note the difference between Eqs. (1) and (2), the slope $f(x_n, y_n)$ in Euler's method becomes $\frac{f(x_n, y_n) + f(x_{n+1}, y^*_{n+1})}{2}$ in improved Euler's method.*

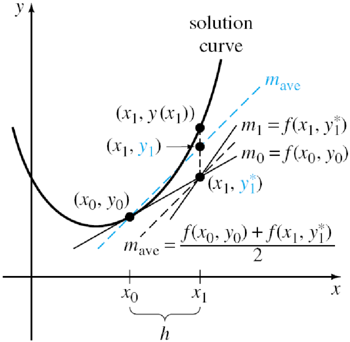
# Improved Euler's Method (2/2)

Improved Euler's method is an example of a **predictor-corrector method**.

The value of $y^*_{n+1}$ given by (3) predicts a value of $y(x_n)$, whereas the value of $y_{n+1}$ defined by formula (2) corrects this estimate.

$$\begin{cases} y_1 = y_0 + h\frac{f(x_0,y_0)+f(x_1,y_1^*)}{2} = y_0 + h\frac{m_0+m_1}{2} \\ y_1^* = y_0 + hf(x_0,y_0) \end{cases}$$

(Compare with Euler's method! Slide 2)



# Example 2: Improved Euler's Method

Use the improved Euler's method to obtain the approximate value of $y(1.5)$ for the solution of the initial-value problem $y' = 2xy,\ y(1) = 1$. Compare the results for $h = 0.1$ and $h = 0.05$. The local truncation error for the improved Euler's method is $O(h^3)$.

# Truncation Errors

In the analysis of local truncation error, we assume the value of $y_n$ is exact in the calculation of $y_{n+1}$. In reality, this is not true since it contains local truncation errors from previous stages.
Thus, the total truncation error in $y_{n+1}$ is an accumulation of the errors in each of the previous steps.
This total error is called *global truncation error*.

Truncation errors:

- Local truncation error of Euler's method: $O(h^2)$.
- Global truncation error of Euler's method: $O(h)$.
- Local truncation error of improved Euler's method: $O(h^3)$.
- Global truncation error of improved Euler's method: $O(h^2)$.

# Runge-Kutta Methods (1/3)

Fundamentally, all **Runge-Kutta methods** are generalizations of the basic Euler's formula $y_{n+1} = y_n + hf(x_n, y_n)$ in that the slope function $f$ is replaced by a *weighted average* of slopes over the interval $x_n \le x \le x_{n+1}$. That is,

$$y_{n+1} = y_n + h(w_1 k_1 + w_2 k_2 + \cdots + w_m k_m) \tag{4}$$

where $w_i, i = 1, 2, \ldots, m$, are constants that generally satisfy $w_1 + w_2 + \cdots + w_m = 1$, and each $k_i, i = 1, 2, \ldots, m$, is the function $f$ evaluated at a selected point $(x, y)$ for which $x_n \le x \le x_{n+1}$. (We shall see that the $k_i$ are defined recursively.) The number $m$ is called the **order** of the method.

# Runge-Kutta Methods (2/3)

Euler's method is said to be a **first-order Runge-Kutta method**. (By taking $m = 1$, $w_1 = 1$, and $k_1 = f(x_n, y_n)$.)

The weighted average is choosen so that Eq. (4) agrees with a Taylor polynomial of degree $m$.

If a function $f(x)$ possesses $k + 1$ derivatives that are continuous on an open interval containing $a$ and $x$, then we can write

$$y(x) = y(a) + y'(a)\frac{x-a}{1!} + y''(a)\frac{(x-a)^2}{2!} + \cdots + y^{(k+1)}(c)\frac{(x-a)^{k+1}}{(k+1)!}$$

where $c \in (a, x)$.

If we replace $a$ by $x_n$ and $x$ by $x_{n+1} = x_n + h$, then

$$y(x_{n+1}) = y(x_n + h) = y(x_n) + hy'(x_n) + \frac{h}{2!}y''(x_n) + \cdots + \frac{h^{k+1}}{(k+1)!}y^{(k+1)}(c)$$

where $c \in (x_n, x_{n+1})$.

# Runge-Kutta Methods (3/3)

If $y(x)$ is a solution of $y' = f(x,y)$ in the case $k = 1$ and the remainder $\frac{1}{2}h^2y''(c)$ is small, then a Taylor ploynomial $y(x_{n+1}) = y(x_n) + hy'(x_n)$ of degree one agrees with the approximation formula of Euler's method

$$y_{n+1} = y_n + hy'_n = y_n + hf(x_n, y_n)$$

# A Fourth-Order Runge-Kutta Method

A **fourth-order Runge-Kutta procedure** consists of finding parameters so that the formula

$$y_{n+1} = y_n + h(w_1 k_1 + w_2 k_2 + w_3 k_3 + w_4 k_4)$$

where

$$\begin{aligned}
k_1 &= f(x_n, y_n) \\
k_2 &= f(x_n + \alpha_1 h, y_n + \beta_1 h k_1) \\
k_3 &= f(x_n + \alpha_2 h, y_n + \beta_2 h k_1 + \beta_3 h k_2) \\
k_4 &= f(x_n + \alpha_3 h, y_n + \beta_4 h k_1 + \beta_5 h k_2 + \beta_6 h k_3)
\end{aligned}$$

agrees with a Taylor polynomial of degree four.

# RK4 Method

The most commonly used set of values for the parameters yields the following result:

$$
\begin{aligned}
y_{n+1} &= y_n + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4) \\
k_1 &= f(x_n, y_n) \\
k_2 &= f(x_n + \frac{1}{2}h, y_n + \frac{1}{2}hk_1) \\
k_3 &= f(x_n + \frac{1}{2}h, y_n + \frac{1}{2}hk_2) \\
k_4 &= f(x_n + h, y_n + hk_3)
\end{aligned}
$$

The above is *the classic* Runge-Kutta method or *the RK4 method*.

# Example 1: RK 4 Method

Use the RK4 method with $h = 0.1$ to obtain an approximation to $y(1.5)$ for the solution of $y' = 2xy,\ y(1) = 1$.

# Homework

- Exercises 9.1: 3, 10.
- Exercises 9.2: 5, 12.