Differential Equations

Lecture Set 11
Orthogonal Functions and Fourier Series

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Inner Product

If $\mathbf{u}$ and $\mathbf{v}$ are two vectors in 3-space, then the inner product $(\mathbf{u}, \mathbf{v})$ possesses the following properties:

- $(\mathbf{u}, \mathbf{v}) = (\mathbf{v}, \mathbf{u})$,
- $(k\mathbf{u}, \mathbf{v}) = k(\mathbf{u}, \mathbf{v})$, k a scalar,
- $(\mathbf{u}, \mathbf{u}) = 0$ if $\mathbf{u} = 0$ and $(\mathbf{u}, \mathbf{u}) > 0$ if $\mathbf{u} \neq 0$,
- $(\mathbf{u} + \mathbf{v}, \mathbf{w}) = (\mathbf{u}, \mathbf{w}) + (\mathbf{v}, \mathbf{w})$.

Definition (11.1.1: Inner Product of Functions)

The **inner product** of two functions f_1 and f_2 on an interval $[a, b]$ is the number

$$(f_1, f_2) = \int_a^b f_1(x) f_2(x) dx$$

Orthogonal Function

Definition (11.1.2: Orthogonal Function)

Two functions f_1 and f_2 are **orthogonal** on an interval $[a, b]$ if

$$(f_1, f_2) = \int_a^b f_1(x) f_2(x) dx = 0$$

Example

The functions $f_1(x) = x^2$ and $f_2(x) = x^3$ are orthogonal on the interval $[-1, 1]$, since

$$\int_{-1}^{1} x^2 \cdot x^3 dx = \int_{-1}^{1} x^5 dx = 0$$

Orthogonal Set

Definition (11.3: Orthogonal Set)

A set of real-valued functions $\{\phi_0(x), \phi_1(x), \phi_2(x), \ldots\}$ is said to be **orthogonal** on an interval $[a, b]$ if

$$(\phi_m, \phi_n) = \int_a^b \phi_m(x)\phi_n(x)dx = 0, \quad m \neq n$$

Orthonormal Sets

The norm, or length $\|\mathbf{u}\|$, of a vector $\mathbf{u}$ can be expressed in terms of the inner product. The expression $(\mathbf{u}, \mathbf{u}) = \|\mathbf{u}\|^2$ is called the square norm, and so the norm is $\|\mathbf{u}\| = \sqrt{(\mathbf{u}, \mathbf{u})}$.

The **square norm** of a function ϕ_n is $\|\phi_n(x)\|^2 = (\phi_n, \phi_n)$, and so the **norm**, or its generalized length, is $\|\phi(x)\| = \sqrt{(\phi_n, \phi_n)}$. They can be written as

$$\|\phi_n(x)\|^2 = \int_a^b \phi_n^2(x)dx \qquad \text{and} \qquad \|\phi_n(x)\| = \sqrt{\int_a^b \phi_n^2(x)dx}$$

If $\{\phi_n(x)\}$ is an orthogonal set of functions on the interval $[a, b]$ with the property that $\|\phi_n(x)\| = 1$ for $n = 0, 1, 2, \ldots$, then $\{\phi_n(x)\}$ is said to be an **orthonormal set** on the interval.

Example 1: Orthogonal Set of Functions

Show that the set $\{1, \cos x, \cos 2x, \ldots\}$ is orthogonal on the interval $[-\pi, \pi]$.

Example 2: Norms

Find the norm of each function in the orthogonal set given in the previous example.

Normalization of Orthogonal Set

Any orthogonal set of nonzero functions $\{\phi_n(x)\}, n = 0, 1, 2, \ldots$ can be *normalized* by dividing each function by its norm.

For example, the set

$$\{\frac{1}{\sqrt{2\pi}}, \frac{\cos x}{\sqrt{\pi}}, \frac{\cos 2x}{\sqrt{\pi}}, \ldots\}$$

is orthonormal on the interval $[-\pi, \pi]$.

Suppose $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are three mutually orthogonal nonzero vectors in 3-space. Then any 3-D vector can be written as

$$\mathbf{u} = \frac{(\mathbf{u}, \mathbf{v}_1)}{||\mathbf{v}_1||^2}\mathbf{v}_1 + \frac{(\mathbf{u}, \mathbf{v}_2)}{||\mathbf{v}_2||^2}\mathbf{v}_2 + \frac{(\mathbf{u}, \mathbf{v}_3)}{||\mathbf{v}_3||^2}\mathbf{v}_3 = \sum_{n=1}^{3} \frac{(\mathbf{u}, \mathbf{v}_n)}{||\mathbf{v}_n||^2}\mathbf{v}_n$$

Orthogonal Series Expansion (1/2)

If $\{\phi_n(x)\}$ is orthogonal w.r.t. a weight function $w(x)$ on the interval $[a, b]$, then multiplying

$$f(x) = c_0\phi_0(x) + c_1\phi_1(x) + c_2\phi_2(x) + \cdots + c_n\phi_n(x) + \cdots$$

by $w(x)\phi_n(x)$ and integrating yields[1]

$$c_n = \frac{\int_a^b f(x)w(x)\phi_n(x)dx}{\|\phi_n(x)\|^2} \tag{1}$$

where

$$\|\phi_n(x)\|^2 = \int_a^b w(x)\phi_n^2(x)dx$$

1

$$\int_a^b f(x)w(x)\phi_n(x)dx = c_n \int_a^b w(x)\phi_n^2(x)dx = c_n\|\phi_n(x)\|^2$$

Orthogonal Series Expansion (2/2)

The series

$$f(x) = \sum_{n=0}^{\infty} c_n \phi_n(x)$$

with coefficients given by Eq. (1), i.e.,

$$c_n = \frac{\int_a^b f(x) w(x) \phi_n(x) dx}{\|\phi_n(x)\|^2}$$

is said to be an **orthogonal series expansion** of f or a **generalized Fourier series.**

Orthogonal Set/Weight Function

Definition (11.1.4: Orthogonal Set/Weight Function)

A set of real-valued functions $\{\phi_0(x), \phi_1(x), \phi_2(x), \ldots\}$ is said to be **orthogonal with respect to a weight function** $w(x)$ on an interval $[a, b]$ if

$$\int_a^b w(x)\phi_m(x)\phi_n(x)dx = 0, \quad m \neq n$$

A Trigonometric Series (1/2)

Suppose that f is a function defined on the interval $[-\pi, \pi]$ and can be expanded in an orthogonal series consisting of the trigonometric functions in the orthogonal set

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi}{p} x + b_n \sin \frac{n\pi}{p} x \right) \tag{2}$$

Integrating both side of Eq. (2) from $-p$ to p gives[2]

$$a_0 = \frac{1}{p} \int_{-p}^{p} f(x) dx$$

2

$$\int_{-p}^{p} f(x) dx = \frac{a_0}{2} \cdot 2p + 0$$

A Trigonometric Series (2/2)

Multiplying Eq. (2) by $\cos(m\pi x/p)$ and taking integration yields

$$a_n = \frac{1}{p}\int_{-p}^{p} f(x)\cos\frac{n\pi}{p}xdx$$

by orthogonality.

Similarly, multiplying Eq. (2) by $\sin(m\pi x/p)$ and taking integration yields

$$b_n = \frac{1}{p}\int_{-p}^{p} f(x)\sin\frac{n\pi}{p}xdx$$

by orthogonality.

Definition (11.2.1: Fourier Series)

The **Fourier series** of a function f defined on the interval $(-p, p)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi}{p} x + b_n \sin \frac{n\pi}{p} x \right)$$

where

$$a_0 = \frac{1}{p} \int_{-p}^{p} f(x) dx$$

$$a_n = \frac{1}{p} \int_{-p}^{p} f(x) \cos \frac{n\pi}{p} x dx$$

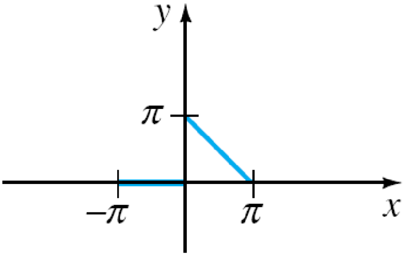
$$b_n = \frac{1}{p} \int_{-p}^{p} f(x) \sin \frac{n\pi}{p} x dx$$

Example 1: Expansion in Fourier Series

Expand

$$f(x) = \begin{cases} 0, & -\pi < x < 0 \\ \pi - x, & 0 \le x < \pi \end{cases}$$

in a Fourier series.



Conditions for Convergence

Theorem (11.2.1: Conditions for Convergence)

Let f and f' be piecewise continuous on the interval $(-p, p)$; that is, let f and f' be continuous except at a finite number of points in the interval and have only finite discontinuities at these points. Then the Fourier series of f on the interval converges to $f(x)$ at a point of continuity. At a point of discontinuity the Fourier series converges to the average

$$\frac{f(x+) + f(x-)}{2}$$

where $f(x+)$ and $f(x-)$ denote the limit of f at x from the right and from the left, respectively.

Example 2: Convergence of a Point of Discontinuity

Show that the function given in the previous example converges at any point on the interval $(-\pi, \pi)$.

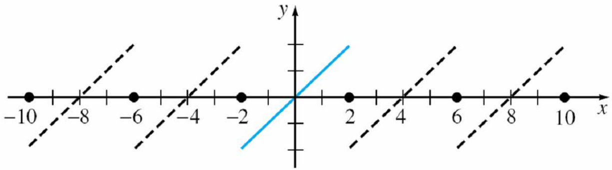
Periodic Extension

A Fourier series not only represents the function on the interval $(-p, p)$, but also gives the **periodic extension** of f outside this interval.

When f is piecewise continuous and the right- and left-hand derivatives exist at $x = -p$ and $x = p$, respectively, then the series Eq. (2) converges to the average

$$\frac{f(p-) + f(p+)}{2}$$

at these endpoints and to this value extended periodically to $\pm 3p$, $\pm 5p$, $\pm 7p$, etc.



Even and Odd Functions

A function f is said to be **even** if $f(-x) = f(x)$.

A function f is said to be **odd** if $f(-x) = -f(x)$.

For example, $f(x) = x^2$ is even and $f(x) = x^3$ is odd; $f(x) = \cos x$ is even and $f(x) = \sin x$ is odd; $f(x) = e^x$ is neither odd nor even.

Properties of Even/Odd Functions

Theorem (11.3.1: Properties of Even/Odd Functions)

- *The product of two even functions is even.*
- *The product of two odd functions is even.*
- *The product of an even function and an odd function is odd.*
- *The sum (difference) of two even functions is even.*
- *The sum (difference) of two odd functions is odd.*
- *If f is even, then $\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx$.*
- *If f is odd, then $\int_{-a}^{a} f(x)dx = 0$.*

Fourier Cosine Series

Definition (11.3.1: Fourier Cosine Series)

The Fourier series of an even function on the interval $(-p, p)$ is the **cosine series**

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi}{p} x$$

where

$$a_0 = \frac{2}{p} \int_0^p f(x) dx \qquad \text{and} \qquad a_n = \frac{2}{p} \int_0^p f(x) \cos \frac{n\pi}{p} x dx$$

Fourier Sine Series

Definition (11.3.1: Fourier Sine Series)

The Fourier series of an odd function on the interval $(-p, p)$ is the **sine series**

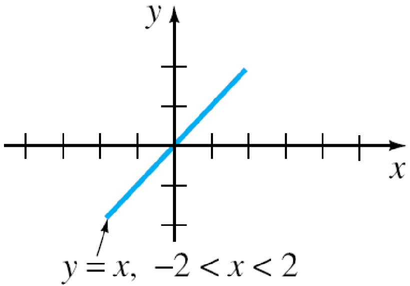
$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{p} x$$

where

$$b_n = \frac{2}{p} \int_0^p f(x) \sin \frac{n\pi}{p} x dx$$

Example 1: Expansion in a Sine Series

Expand $f(x) = x, \ -2 < x < 2$ in a Fourier series.

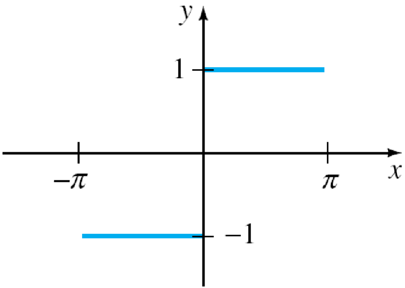


Example 2: Expansion in a Sine Series

Expand the function

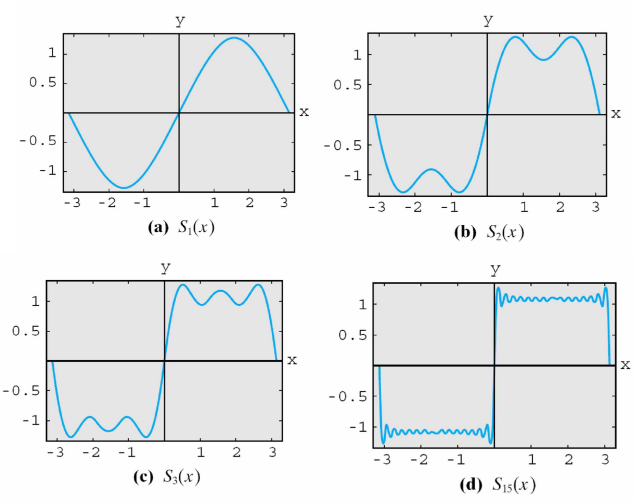
$$f(x) = \begin{cases} -1, & -\pi < x < \pi \\ 1, & 0 \le x < \pi \end{cases}$$

in a Fourier series.



Gibbs Phenomenon

If we process the function term-by-term...



Eigenvalues and Eigenfunctions (1/4)

- Orthogonal functions arise in the solution of differential equations.
- An orthogonal set of functions can be generated by solving a certain kind of two-point bounrady-value problem involving a linear 2nd-order DE containing a parameter λ.

Eigenvalues and Eigenfunctions (2/4)

Example

The boundary-value problem

$$y'' + \lambda y = 0, \quad y(0) = 0, \quad y(L) = 0 \tag{3}$$

possesses nontrivial solutions only when the parameter λ took on the values $\lambda_n = n^2\pi^2/L^2$, $n = 1, 2, 3, \ldots$, called **eigenvalues.**

The corresponding nontrivial solutions $y_n = c_2 \sin(n\pi x/L)$, or simply $y_n = \sin(n\pi x/L)$, are called the **eigenfunctions** of the problem. (Check page 215, Example 2, Section 5.2, on the textbook.)

Eigenvalues and Eigenfunctions (3/4)

Example

The boundary-value problem

$$y'' - 2y = 0, \quad y(0) = 0, \quad y(L) = 0$$

only possesses trivial solution $y = 0$ since $\lambda = -2$ is not an eigenvalue.

Eigenvalues and Eigenfunctions (4/4)

Example

The boundary-value problem

$$y'' + \frac{9\pi^2}{L^2}y = 0, \quad y(0) = 0, \quad y(L) = 0$$

possesses a nontrivial solution $y_3 = \sin(3\pi x/L)$ since $\lambda = 9\pi^2/L^2$ is an eigenvalue. Furthermore, $y_3 = \sin(3\pi x/L)$ is an eigenfunction.

Remark

The set of trigonometric functions generated by this BVP, i.e., $\{\sin(n\pi x/L)\}, n = 1, 2, 3, \ldots$, is an orthogonal set of functions on the interval $[0, L]$ and is used as the basis for the Fourier sine series.

Example 1: Eigenvalues and Eigenfunctions

Solve the boundary-value problem

$$y'' + \lambda y = 0, \quad y'(0) = 0, \quad y'(L) = 0 \tag{4}$$

Regular Sturm-Liouville Problem (1/3)

The problems in Eqs. (3) and (4), i.e.

$$y'' + \lambda y = 0, \quad y(0) = 0, \quad y(L) = 0$$

and

$$y'' + \lambda y = 0, \quad y'(0) = 0, \quad y'(L) = 0$$

are special cases of an important general two-point BVP.

Let p, q, r and r' be real-valued functions continuous on an interval $[a, b]$, and let $r(x) > 0$ and $p(x) > 0$ for every x in the interval. Then

$$Solve: \quad \frac{d}{dx}[r(x)y'] + (q(x) + \lambda p(x))y = 0 \tag{5}$$

$$Subject\ to: \quad \begin{cases} A_1 y(a) + B_1 y'(a) = 0 \\ A_2 y(b) + B_2 y'(b) = 0 \end{cases} \tag{6}$$

is said to be a **regular Sturm-Liouville problem**.

Regular Sturm-Liouville Problem (2/3)

The BVPs in Eqs. (3) and (4), i.e.

$$y'' + \lambda y = 0, \quad y(0) = 0, \quad y(L) = 0$$

and

$$y'' + \lambda y = 0, \quad y'(0) = 0, \quad y'(L) = 0$$

are regular Sturm-Liuoville problems.

The DE (5) is linear and homogeneous. The boundary conditions in Eqs. (6) are also homogeneous.

A boundary condition such as $Ay(b) + By'(b) = C$, where C is a nonzero constant, is nonhomogeneous.

Regular Sturm-Liouville Problem (3/3)

A BVP that consists of a homogeneous linear DE and homogeneous BCs is said to be a homogeneous BVP; otherwise, it is nonhomogeneous.

The BCs Eqs. (6) are referred to as **separated** since each condition involves only a single boundary point.

$$\begin{cases} A_1 y(a) + B_1 y'(a) = 0 \\ A_2 y(b) + B_2 y'(b) = 0 \end{cases}$$

Because a regular Sturm-Liouville problem is a homogeneous BVP; it always possess the trivial solution $y = 0$.

Properties of Regular S-L Problem

Theorem (11.4.1: Properties of Regular S-L Problem)

(a) *There exist an infinite number of real eigenvalues that can be arranged in increasing order* $\lambda_1 < \lambda_2 < \lambda_3 < \cdots < \lambda_n < \cdots$ *such that* $\lambda_n \to \infty$ *as* $n \to \infty$.

(b) *For each eigenvalue there is only one eigenfunction (except for nonzero constant multiples).*

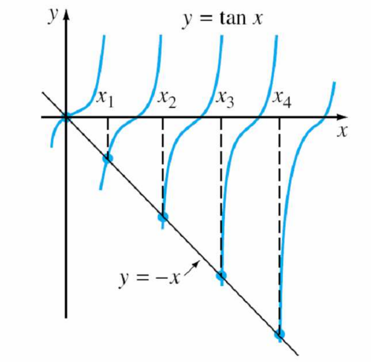
(c) *Eigenfunctions corresponding to different eigenvalues are linearly independent.*

(d) *The set of eigenfunctions corresponding to the set of eigenvalues is orthogonal with respect to the weight function* $p(x)$ *on the interval* $[a, b]$.

Example 2: A Regular Sturm-Liouville Problem

Solve the boundary-value problem

$$y'' + \lambda y = 0, \quad y(0) = 0, \quad y(1) + y'(1) = 0$$



Homework

- Exercises 11.1: 4, 9.
- Exercises 11.2: 6, 11.
- Exercises 11.3: 8, 15, 28, 35.
- Exercises 11.4: 1.