Differential Equations

Lecture Set 12
Boundary-Value Problems in Rectangular Coordinates

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Linear Partial Differential Equation

- Let u denote the dependent variable and let x and y denote the independent variables, then the general form of a **linear second-order partial differential equation** is given by

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = G$$

  where the coefficients $A, B, C, \ldots, G$ are functions of x and y.
- When $G(x, y) = 0$, the equation is said to be **homogeneous**; otherwise, it is **nonhomogeneous.** For example, the linear equations

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{and} \quad \frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial y} = xy$$

  are homogeneous and nonhomogeneous, respectively.

Solution of a PDE

A **solution** of a linear PDE is a function $u(x, y)$ of two independent variables that possesses all partial derivatives occurring in the equation and that satisfies the equation in some region of the xy-plane.

Finding a general solution of a linear PDE is difficult and not very useful in applications.
Thus we will find only *particular solutions* of some of the more important linear PDEs.

Separation of Variables

In the method of **separation of variables**, we seek a particular solution of the form of a *product* of a function of x and a function of y:

$$u(x,y) = X(x)Y(y)$$

With this assumption it is *sometimes* possible to reduce a linear PDE in two variables to two ODEs. To this end we note that

$$\frac{\partial u}{\partial x} = X'Y, \quad \frac{\partial u}{\partial y} = XY', \quad \frac{\partial^2 u}{\partial x^2} = X''Y, \quad \frac{\partial^2 u}{\partial y^2} = XY''$$

where the primes denote ordinary differentiation.

Example 1: Separable of Variables

Find product solutions of

$$\frac{\partial^2 u}{\partial x^2} = 4\frac{\partial u}{\partial y} \tag{1}$$

Superposition Principle

Theorem (12.1.1: Superposition Principle)

If $u_1, u_2, \cdots, u_k$ are solutions of a homogeneous linear PDE, then the linear combination

$$u = c_1 u_1 + c_2 u_2 + \cdots + c_k u_k$$

where $c_i, i = 1, 2, \cdots, k$, are constants, is also a solution.

Classification of Equations

Definition (12.1.1: Classification of Equations)

The linear 2nd-order PDE

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = G$$

where the coefficients A, B, C, D, E, and F are constants, is said to be

hyperbolic if	$B^2 - 4AC > 0$
parabolic if	$B^2 - 4AC = 0$
elliptic if	$B^2 - 4AC < 0$

Example 2: Classifying Linear Second-Order PDEs

Classify the following equations

(a) $3\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial y}$

(b) $\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial y^2}$

(c) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

The PDEs

$$k\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, \quad k > 0$$

$$a^2\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

are known, respectively, as the **one-dimensional heat equation**, the **one-dimensional wave equation**, and the **two-dimensional form of Laplace's equation.**

Boundary-Value Problems

Problems such as

$$\begin{aligned}
&\textit{Solve}: \quad a^2\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2},\ 0 < x < L,\ t > 0 \\
&\textit{Subject to}: \quad \begin{array}{ll} \text{(BC)} & u(0,t) = 0,\ u(L,t) = 0,\ t > 0 \\ \text{(IC)} & u(x,0) = f(x),\ \frac{\partial u}{\partial t}\Big|_{t=0} = g(x),\ 0 < x < L \end{array}
\end{aligned}$$

and

$$\begin{aligned}
&\textit{Solve}: \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,\ 0 < x < a,\ 0 < y < b \\
&\textit{Subject to}: \ \text{(BC)} \begin{cases} \frac{\partial u}{\partial x}\Big|_{x=0} = 0,\ \frac{\partial u}{\partial x}\Big|_{x=a} = 0,\ 0 < y < b \\ u(x,0) = 0,\ u(x,b) = f(x),\ 0 < x < a \end{cases}
\end{aligned}$$

are called **boundary-value problems.**