Differential Equations

Lecture Set 14
Integral Transform Method

林惠勇
Huei-Yung Lin

lin@ee.ccu.edu.tw

Robot Vision Lab
Department of Electrical Engineering
National Chung Cheng University
Chiayi 621, Taiwan

Fourier Series to Fourier Integral (1/2)

Suppose a function f is defined on $(-p, p)$. Then from Definition 11.2.1, the Fourier series of f on the interval is

$$\begin{aligned} f(x) &= \frac{1}{2p}\int_{-p}^{p} f(t)dt + \frac{1}{p}\sum_{n=1}^{\infty}\left[\left(\int_{-p}^{p} f(t)\cos\frac{n\pi}{p}tdt\right)\cos\frac{n\pi}{p}x \right. \\ &\qquad \left. + \left(\int_{-p}^{p} f(t)\sin\frac{n\pi}{p}tdt\right)\sin\frac{n\pi}{p}x\right] \qquad (1) \\ &= \frac{1}{2\pi}\left(\int_{-p}^{p} f(t)dt\right)\Delta\alpha + \frac{1}{\pi}\sum_{n=1}^{\infty}\left[\left(\int_{-p}^{p} f(t)\cos\alpha_n tdt\right)\cos\alpha_n x \right. \\ &\qquad \left. + \left(\int_{-p}^{p} f(t)\sin\alpha_n tdt\right)\sin\alpha_n x\right]\Delta\alpha \qquad (2) \end{aligned}$$

if we let $\alpha_n = n\pi/p, \Delta\alpha = \alpha_{n+1} - \alpha_n = \pi/p$. $\left(\Delta\alpha = \pi/p \Rightarrow \frac{1}{2p} = \frac{\Delta\alpha}{2\pi}\right)$

Fourier Series to Fourier Integral (2/2)

Now, if we let $p \to \infty$, i.e., $\Delta\alpha \to 0$, then Eq. (2) becomes

$$f(x) = \frac{1}{\pi}\int_0^\infty \left[\left(\int_{-\infty}^{\infty} f(t)\cos\alpha t dt\right)\cos\alpha x + \left(\int_{-\infty}^{\infty} f(t)\sin\alpha t dt\right)\sin\alpha x\right] d\alpha \qquad (3)$$

since $\lim_{\Delta\alpha\to 0}\sum_{n=1}^{\infty} F(\alpha_n)\Delta\alpha$ is suggestive of the definition of the integral $\int_0^\infty F(\alpha)d\alpha$, and the limit of the first term in Eq. (2) goes to zero.[1]

Eq. (3) is called the **Fourier integral** of f on $(-\infty, \infty)$. The basic structure of the Fourier integral is reminiscent of that of a Fourier series.

[1] If $\int_{-\infty}^{\infty} f(x)dx$ exists (and bounded!), then $[\int_{-\infty}^{\infty} f(x)dx]\Delta\alpha \to 0$. $\surd$

Definition (14.3.1: Fourier Integral)

The **Fourier Integral** of a function f defined on the interval $(-\infty, \infty)$ is given by

$$f(x) = \frac{1}{\pi} \int_0^\infty [A(\alpha) \cos \alpha x + B(\alpha) \sin \alpha x] \, d\alpha \tag{4}$$

where

$$A(\alpha) = \int_{-\infty}^{\infty} f(x) \cos \alpha x dx$$

$$B(\alpha) = \int_{-\infty}^{\infty} f(x) \sin \alpha x dx$$

Conditions for Convergence

Theorem (14.3.1: Conditions for Convergence)

Let f and f' be piecewise continuous on every finite interval, and let f be absolutely integrable on $(-\infty, \infty)$. The the Fourier integral of f on the interval converges to $f(x)$ at a point of continuity. At a point of discontinuity the Fourier integral will converge to the average

$$\frac{f(x+) + f(x-)}{2}$$

where $f(x+)$ and $f(x-)$ denote the limit of f at x from the right and from the left, respectively.

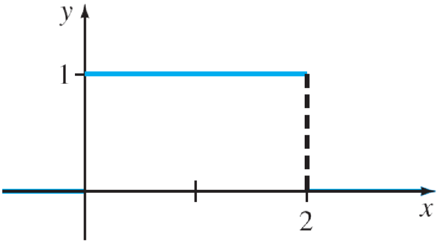
Remark

"Absolutely integrable" means that the integral $\int_{-\infty}^{\infty} |f(x)|dx$ converges.

Example 1: Fourier Integral Representation

Find the Fourier integral representation of the function

$$f(x) = \begin{cases} 0, & x < 0 \\ 1, & 0 < x < 2 \\ 0, & x > 2 \end{cases}$$



Remark

From the previous example

$$f(x) = \frac{2}{\pi} \int_0^\infty \frac{\sin\alpha \cos\alpha(x-1)}{\alpha} d\alpha \quad (5)$$

is the Fourier integral of

$$f(x) = \begin{cases} 0, & x < 0 \\ 1, & 0 < x < 2 \\ 0, & x > 2 \end{cases}$$

Since Eq. (5) converges to $f(1) = 1$ by Theorem 14.1, we have

$$\frac{2}{\pi} \int_0^\infty \frac{\sin\alpha}{\alpha} d\alpha = 1 \quad \text{or} \quad \int_0^\infty \frac{\sin\alpha}{\alpha} d\alpha = \frac{\pi}{2}$$

Cosine and Sine Integrals

When f is an even function on the interval $(-\infty, \infty)$, then the product $f(x)\cos\alpha x$ is also an even function whereas $f(x)\sin\alpha x$ is an odd function. By properties (g) and (f) of Theorem 11.2, Eq. (4) becomes

$$f(x) = \frac{2}{\pi}\int_0^\infty \left(\int_0^\infty f(x)\cos\alpha x dx\right)\cos\alpha x d\alpha$$

Similarly, when f is an odd function on $(-\infty, \infty)$, product $f(x)\cos\alpha x$ and $f(x)\sin\alpha x$ are odd and even functions, respectively. Thus, Eq. (4) becomes

$$f(x) = \frac{2}{\pi}\int_0^\infty \left(\int_0^\infty f(x)\sin\alpha x dx\right)\sin\alpha x d\alpha$$

Fourier Cosine and Sine Integrals

Definition (14.3.2: Fourier Cosine and Sine Integrals)

(i) The Fourier integral of an even function on the interval $(-\infty, \infty)$ is the **cosine integral**

$$f(x) = \frac{2}{\pi} \int_0^\infty A(\alpha) \cos \alpha x d\alpha$$

where

$$A(\alpha) = \int_0^\infty f(x) \cos \alpha x dx$$

Fourier Cosine and Sine Integrals

Definition (14.3.2: Fourier Cosine and Sine Integrals)

(ii) The Fourier integral of an odd function on the interval $(-\infty, \infty)$ is the **sine integral**

$$f(x) = \frac{2}{\pi} \int_0^\infty B(\alpha) \sin \alpha x d\alpha$$

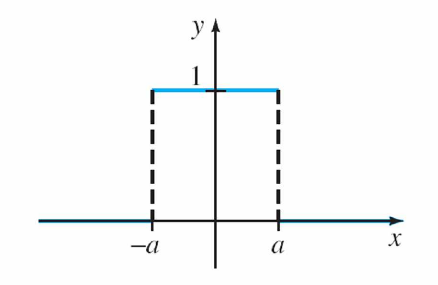
where

$$B(\alpha) = \int_0^\infty f(x) \sin \alpha x dx$$

Example 2: Cosine Integral Representation

Find the Fourier integral representation of the function

$$f(x) = \begin{cases} 1, & |x| < a \\ 0, & |x| > a \end{cases}$$



Example 3: Cosine and Sine Integral Representation

Represent $f(x) = e^{-x}, x > 0$

(a) by a cosine integral,

(b) by a sine integral.

Complex Form

The Fourier integral possesses an equivalent **complex form**, or **exponential form** as

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} C(\alpha) e^{-i\alpha x} d\alpha$$

where

$$C(\alpha) = \int_{-\infty}^{\infty} f(x) e^{i\alpha x} dx$$

Transform Pairs

Integral transforms appear in **transform pairs.** If $f(x)$ is transformed into $F(\alpha)$ by an **integral transform**

$$F(\alpha) = \int_a^b f(x)K(\alpha, x)dx$$

then the function f can be recovered by another integral transform

$$f(x) = \int_c^d F(\alpha)H(\alpha, x)d\alpha$$

called the **inverse transform.** The functions K and H in the integrands are called the **kernels** of their respective transforms.

We identify $K(s,t) = e^{-st}$ as the kernel of the Laplace transform and $H(s,t) = e^{st}/2\pi i$ as the kernel of the inverse Laplace transform.

Fourier Transform Pairs

Definition (14.4.1: Fourier Transform Pairs)

Fourier transform: $\mathscr{F}\{f(x)\} = \int_{-\infty}^{\infty} f(x)e^{i\alpha x}dx = F(\alpha)$

Inverse Fourier transform: $\mathscr{F}^{-1}\{F(\alpha)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\alpha)e^{-i\alpha x}d\alpha = f(x)$

Fourier sine transform: $\mathscr{F}\{f(x)\} = \int_{0}^{\infty} f(x)\sin\alpha x dx = F(\alpha)$

Inverse Fourier sine transform: $\mathscr{F}^{-1}\{F(\alpha)\} = \frac{2}{\pi}\int_{0}^{\infty} F(\alpha)\sin\alpha x d\alpha = f(x)$

Fourier cosine transform: $\mathscr{F}\{f(x)\} = \int_{0}^{\infty} f(x)\cos\alpha x dx = F(\alpha)$

Inverse Fourier cosine transform: $\mathscr{F}^{-1}\{F(\alpha)\} = \frac{2}{\pi}\int_{0}^{\infty} F(\alpha)\cos\alpha x d\alpha = f(x)$

Operational Properties

Fourier Transform

$$\mathscr{F}\left\{f'(x)\right\} = -i\alpha F(\alpha)$$
$$\mathscr{F}\left\{f''(x)\right\} = -\alpha^2 F(\alpha)$$
$$\mathscr{F}_s\left\{f'(x)\right\} = -\alpha \mathscr{F}_c\left\{f(x)\right\}$$
$$\mathscr{F}_c\left\{f'(x)\right\} = \alpha \mathscr{F}_s\left\{f(x)\right\} - f(0)$$

Fourier Sine Transform

$$\mathscr{F}_s\left\{f''(x)\right\} = -\alpha^2 F(\alpha) + \alpha f(0)$$

Fourier Cosine Transform

$$\mathscr{F}_c\left\{f''(x)\right\} = -\alpha^2 F(\alpha) + f'(0)$$

Example 1: Using the Fourier Transform

Solve the heat equation

$$k\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, \quad -\infty < x < \infty,\ t > 0$$

subject to

$$u(x,0) = f(x), \quad \text{where} \quad f(x) = \begin{cases} u_0, & |x| < 1 \\ 0, & |x| > 1 \end{cases}$$